

EQUIVARIANT ORBIFOLD STRUCTURES ON THE PROJECTIVE LINE AND INTEGRABLE HIERARCHIES

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ABSTRACT. Let $\mathbb{CP}^1_{k,m}$ be the orbifold structure on \mathbb{CP}^1 obtained via uniformizing the neighborhoods of 0 and ∞ respectively by $z \mapsto z^k$ and $w \mapsto w^m$. The diagonal action of the torus $T = (S^1)^2$ on \mathbb{CP}^1 induces naturally an action on the orbifold $\mathbb{CP}^1_{k,m}$. In this paper we prove that if k and m are co-prime then Givental's prediction of the equivariant total descendent Gromov-Witten potential of $\mathbb{CP}^1_{k,m}$ satisfies certain Hirota Quadratic Equations (HQE for short). We also show that after an appropriate change of the variables, similar to Getzler's change in the equivariant Gromov-Witten theory of \mathbb{CP}^1 , the HQE turn into the HQE of the 2-Toda hierarchy, i.e., the Gromov-Witten potential of $\mathbb{CP}^1_{k,m}$ is a tau-function of the 2-Toda hierarchy. More precisely, we obtain a sequence of tau-functions of the 2-Toda hierarchy from the descendent potential via some translations. The later condition, that all tau-functions in the sequence are obtained from a single one via translations, imposes a serious constraint on the solution of the 2-Toda hierarchy. Our theorem leads to the discovery of a new integrable hierarchy (we suggest to be called the Equivariant Bi-graded Toda Hierarchy), obtained from the 2-Toda hierarchy via a reduction similar to the one in [13]. We conjecture that this new hierarchy governs, i.e., uniquely determines, the equivariant Gromov-Witten invariants of $\mathbb{CP}^1_{k,m}$.

1. INTRODUCTION

Let $\mathbb{CP}^1_{k,m}$ be the orbifold structure on \mathbb{CP}^1 obtained via uniformizing the neighborhoods of 0 and ∞ respectively by $z \mapsto z^k$ and $w \mapsto w^m$. This uniformization induces naturally an orbifold structure on the hyperplane class bundle, such that the cyclic groups \mathbb{Z}_k and \mathbb{Z}_m act trivially on the corresponding fibers. The resulting orbifold bundle is denoted $\mathcal{O}^{\text{unif}}(1)$.

Let $T = S^1 \times S^1$ and denote by ν_0 and ν_1 the characters of the representation dual to the standard representation of T in \mathbb{C}^2 . The T -equivariant cohomology of a point is naturally identified with $\mathbb{C}[\nu_0, \nu_1]$. Furthermore, the diagonal action of T on \mathbb{C}^2 induces a T -action on $\mathbb{CP}^1 = (\mathbb{C}^2 - \{0\})/\mathbb{C}^*$ and the later

Date: February 1, 2008.

2000 Math. Subj. Class. 14N35, 17B69, 32S30.

Key words and phrases. oscillating integrals, Frobenius structure, orbifold quantum cohomology, bosonic Fock space, vertex operators, Hirota quadratic (bilinear) equations.

naturally induces a T -action on the orbifold $\mathbb{CP}_{k,m}^1$. We also equip the bundle $\mathcal{O}^{\text{unif}}(1)$ with a T -action in such a way that the corresponding characters on the fibers of $\mathcal{O}^{\text{unif}}(1)$ at 0 and ∞ are respectively ν_0 and ν_1 .

The equivariant orbifold cohomology H of $\mathbb{CP}_{k,m}^1$ is by definition the equivariant cohomology of its inertia orbifold:

$$I\mathbb{CP}_{k,m}^1 = \mathbb{CP}_{k,m}^1 \sqcup \bigsqcup_{i=1}^{k-1} [\text{pt}/\mathbb{Z}_k] \sqcup \bigsqcup_{j=1}^{m-1} [\text{pt}/\mathbb{Z}_m],$$

where the orbifolds $[\text{pt}/\mathbb{Z}_k]$ and $[\text{pt}/\mathbb{Z}_m]$ are called *twisted sectors* and the torus T acts trivially on them. We fix a basis in H :

$$\begin{aligned} \mathbf{1}_{i/k}, \quad 1 \leq i \leq k-1, \quad \mathbf{1}_{0/k} &= (p - \nu_1)/(\nu_0 - \nu_1), \\ \mathbf{1}_{j/m}, \quad 1 \leq j \leq m-1, \quad \mathbf{1}_{0/m} &= (p - \nu_0)/(\nu_1 - \nu_0), \end{aligned}$$

where p is the equivariant 1-st Chern class of $\mathcal{O}^{\text{unif}}(1)$, $\mathbf{1}_{i/k}$ and $\mathbf{1}_{j/m}$ are the units in the cohomologies of the corresponding twisted sectors and the indices i/k and j/m are identified respectively with elements in \mathbb{Z}_k and \mathbb{Z}_m . Finally, let $(\ , \)$ be the equivariant orbifold Poincaré pairing in H :

$$\begin{aligned} (\mathbf{1}_{0/k}, \mathbf{1}_{0/k}) &= 1/(\nu_0 - \nu_1), \quad (\mathbf{1}_{i/k}, \mathbf{1}_{(k-i)/k}) = 1/k, \quad 1 \leq i \leq k-1, \\ (\mathbf{1}_{0/m}, \mathbf{1}_{0/m}) &= 1/(\nu_1 - \nu_0), \quad (\mathbf{1}_{j/m}, \mathbf{1}_{(m-j)/m}) = 1/m, \quad 1 \leq j \leq m-1, \end{aligned}$$

and all other pairs of cohomology classes are orthogonal.

By definition *the total descendent Gromov–Witten potential* of $\mathbb{CP}_{k,m}^1$ is

$$\mathcal{D}(\mathbf{q}) = \exp \left(\sum_{g,n,d} \epsilon^{2g-2} \frac{Q^d}{n!} \int_{[\overline{\mathcal{M}}_{g,n}(\mathbb{CP}_{k,m}^1, d)]^{\text{vir}}} \prod_{a=1}^n \left(\psi_a + \sum_{l=0}^{\infty} \text{ev}_a^*(q_l) \psi_a^l \right) \right),$$

where $\overline{\mathcal{M}}_{g,n}(\mathbb{CP}_{k,m}^1, d)$ is the moduli space of degree $d \in \mathbb{Z}$ stable holomorphic maps f from a genus- g Riemann surface, equipped with n marked orbifold points, $\text{ev}_a : \overline{\mathcal{M}}_{g,n}(\mathbb{CP}_{k,m}^1, d) \rightarrow I\mathbb{CP}_{k,m}^1$ is the evaluation map at the a -th marked point, ψ_a is the equivariant 1-st Chern class of the line bundle on $\overline{\mathcal{M}}_{g,n}(\mathbb{CP}_{k,m}^1, d)$ corresponding to the cotangent line at the a -th marked point, $\mathbf{q} = \sum_{l=0}^{\infty} q_l z^l \in H[z]$, the integrals are performed against the *virtual fundamental classes* $[\overline{\mathcal{M}}_{g,n}(\mathbb{CP}_{k,m}^1, d)]^{\text{vir}}$, and the sum is over all non-negative integers g, n, d for which the moduli space $\overline{\mathcal{M}}_{g,n}(\mathbb{CP}_{k,m}^1, d)$ is non-empty.

The potential \mathcal{D} is identified with an element of a bosonic Fock space \mathcal{B} which by definition is the vector space of functions on $H[z]$ which belong to the formal neighborhood of $-\mathbf{1}z$. Note that $\mathbf{1} = \mathbf{1}_{0/k} + \mathbf{1}_{0/m}$, therefore if we put

$$q_n = \sum_{i=0}^{k-1} q_n^{i/k} \mathbf{1}_{i/k} + \sum_{j=0}^{m-1} q_n^{j/m} \mathbf{1}_{j/m},$$

then \mathcal{B} is the set of formal series in the variables $q_n^{i/k} + \delta_n^1 \delta_{0/k}^{i/k}, q_n^{j/m} + \delta_n^1 \delta_{0/m}^{j/m}$, whose coefficients are formal Laurent series in ϵ . Here we used the Kronecker symbols: $\delta_a^b = 1$ or 0 depending whether $a = b$ or $a \neq b$.

We introduce the following *vertex operators* acting on the Fock space \mathcal{B} :

$$(1.1) \quad \Gamma^\pm = \exp \left(\pm \sum_{n \in \mathbb{Z}} \sum_{i=1}^k \frac{\prod_{l=-\infty}^n (\nu + (-i/k + l)z)}{\prod_{l=-\infty}^0 (\nu + (-i/k + l)z)} \lambda^{-(n+1)k+i} \mathbf{1}_{(k-i)/k} \right)^\wedge$$

where $\nu = (\nu_0 - \nu_1)/k$, and the hat \wedge indicates the following quantization rule. The exponent \mathbf{f}^\pm of Γ^\pm is written as a product of two exponents: the first (left) one contains the summands with $n < 0$ and the second (right) one with $n \geq 0$. Each summand corresponding to $n < 0$ is expanded into a series of z^{-1} . The quantization rule consists of representing the terms $\phi_\alpha(-z)^{-n-1}, n \geq 0$ and $\phi_\alpha z^n, n \geq 0$ respectively by the operators of multiplication by the linear function $-\epsilon^{-1} \sum_\beta \eta_{\alpha\beta} q_n^\beta$ and the differential operator $\epsilon \partial / \partial q_n^\alpha$. Here $\eta_{\alpha\beta} = (\mathbf{1}_\alpha, \mathbf{1}_\beta)$ is the tensor of the Poincaré pairing. Similarly, we introduce the vertex operator $\bar{\Gamma}^\pm$ obtained from Γ^\pm by switching $\nu_0 \leftrightarrow \nu_1$, and $k \leftrightarrow m$.

We say that a vector \mathcal{D} in the Fock space \mathcal{B} satisfies the Hirota quadratic equations (HQE) below if for each pair of integers l and n

$$(1.2) \quad \text{res}_{\lambda=\infty} \left(\lambda^{n-l} \Gamma^- \otimes \Gamma^+ - (Q/\lambda)^{n-l} \bar{\Gamma}^+ \otimes \bar{\Gamma}^- \right) \left(e^{(n+1)\hat{\mathbf{1}}_{0/k} + n\hat{\mathbf{1}}_{0/m}} \otimes e^{l\hat{\mathbf{1}}_{0/k} + (l+1)\hat{\mathbf{1}}_{0/m}} \right) (\mathcal{D} \otimes \mathcal{D}) \frac{d\lambda}{\lambda} = 0.$$

The HQE (1.2) are interpreted as follows. Switch to new variables \mathbf{x} and \mathbf{y} via the substitutions: $\mathbf{q}' = \mathbf{x} + \mathbf{y}$, $\mathbf{q}'' = \mathbf{x} - \mathbf{y}$. The LHS of the HQE expands as a series in \mathbf{y} with coefficients Laurent series in λ^{-1} , whose coefficients are quadratic polynomials in \mathcal{D} , its partial derivatives and their translations. The residue is defined as the coefficient in front of λ^{-1} .

Motivated by Givental's formula of the total descendent potential of a Kähler manifold with semi-simple quantum cohomology, we introduce an element \mathcal{D}^{Fr} of the Fock space of the following type:

$$(1.3) \quad \mathcal{D}^{\text{Fr}} = e^{\text{F}^{(1)}(\tau)} \hat{S}_\tau^{-1} \left(\Psi_\tau R_\tau e^{U_\tau/z} \right)^\wedge \prod_{i=1}^{k+m} \mathcal{D}_{\text{pt}}(\mathbf{q}^i).$$

The different ingredients in this formula will be explained later. For a Kähler manifold equipped with a Hamiltonian torus action whose 0 and 1-dimensional strata are isolated, Givental [14, 15] proved that (1.3) agrees with the equivariant total descendant Gromov-Witten potential. His arguments, based on an ingenious localization analysis, may be extended to orbifolds and, together with some new ingredients, used to prove that (1.3) agrees with the equivariant total descendant orbifold Gromov-Witten potential for a Kähler orbifold

with a Hamiltonian torus action whose 0 and 1-dimensional strata are isolated. Details will be given in [22].

Our goal here is to prove that the conjectural formula (1.3) has some very interesting property which in particular leads to the discovery of a new integrable hierarchy given in terms of HQE by (1.2). Our main result can be stated this way.

Theorem 1.1. *The function \mathcal{D}^{Fr} satisfies (1.2).*

Let y_1, y_2, \dots and $\bar{y}_1, \bar{y}_2, \dots$ be two sequences of time variables related to $q_0^{i/k}, q_1^{i/k}, \dots$ and $q_0^{j/m}, q_1^{j/m}, \dots$ via an upper-triangular linear change defined by the following relations:

$$(1.4) \quad \sum_{n \geq 0} (-w)^{-n-1} \frac{\partial}{\partial q_n^{i/k}} = \sum_{n \geq 0} \frac{g_{i/k}}{\prod_{l=0}^n (\nu - (l + i/k)w)} \frac{\partial}{\partial y_{nk+i}},$$

$$(1.5) \quad \sum_{n \geq 0} (-w)^{-n-1} \frac{\partial}{\partial q_n^{j/m}} = \sum_{n \geq 0} \frac{g_{j/m}}{\prod_{l=0}^n (\bar{\nu} - (l + j/m)w)} \frac{\partial}{\partial \bar{y}_{nm+j}},$$

where $n \geq 0$, $1 \leq i \leq k$, $1 \leq j \leq m$, $\bar{\nu} = (\nu_1 - \nu_0)/m$ and

$$g_\alpha := (\mathbf{1}_\alpha, \mathbf{1}_{-\alpha}), \quad \alpha \in \mathbb{Z}_k \sqcup \mathbb{Z}_m.$$

Theorem 1.2. *Let $\mathcal{D}_n(\mathbf{q}) = Q^{n^2/2} \mathcal{D}^{\text{Fr}}(\mathbf{q} + n\epsilon \mathbf{1})$. Then the changes (1.4)–(1.5) transforms $\{\mathcal{D}_n\}$ into a sequence of tau-functions of the 2-Toda hierarchy.*

Recall that the KdV hierarchy is a reduction of the KP hierarchy which in terms of tau-functions can be described as follows: tau-functions of KdV hierarchy are tau-functions of KP which depend only on odd variables. In our case we have a reduction of the 2-Toda hierarchy which in terms of tau-functions can be described as sequences of tau-functions of 2-Toda obtained from a single function by some translations. In Appendix B we describe what kind of constraint the later condition imposes on the Lax operators of 2-Toda.

Acknowledgments. We are thankful to B. Dubrovin for showing interest in our work and for pointing out that the 2-Toda hierarchy is too big to govern the Gromov-Witten theory of the orbifold $\mathbb{CP}_{k,m}^1$. This made us realize that our HQEs describe a reduction of the 2-Toda hierarchy. Many thanks to the organizers M. Bertola and D. Korotkin of the conference “Short program on Moduli spaces of Riemann surfaces and related topics”, held in Montreal, Canada during June 3 – 15, 2007, where the first author was given the opportunity to lecture on some of the results in this article. The second author is grateful to Institut Mittag-Leffler (Djursholm, Sweden) for hospitality and support during his visit to the program “moduli spaces”.

2. PROOF OF THEOREM 1.2

Let $h_l(x_1, \dots, x_n)$ and $e_l(x_1, \dots, x_n)$ be the symmetric polynomials of degree l defined as follows:

$$\begin{aligned} \prod_{i=1}^n (1 + tx_i) &= \sum_{l \geq 0} t^l e_l(x_1, \dots, x_n), \\ \prod_{i=1}^n \frac{1}{1 + tx_i} &= \sum_{l \geq 0} t^l h_l(x_1, \dots, x_n). \end{aligned}$$

To avoid cumbersome notations we put

$$\delta_{kN+i} := \frac{g_{i/k}}{(N + i/k)!} \frac{\partial}{\partial y_{kN+i}} \quad \bar{\delta}_{mN+j} := \frac{g_{j/m}}{(N + j/m)!} \frac{\partial}{\partial \bar{y}_{mN+j}},$$

where $N \geq 0$, $1 \leq i \leq k$, $1 \leq j \leq m$, and for a positive real number $\alpha \notin \mathbb{Z}$ we put $\alpha! = \{\alpha\}(\{\alpha\} + 1) \dots \alpha$ where $\{\alpha\}$ is the fractional part of α . Note that the change of variables can be written as follows

$$\begin{aligned} \frac{\partial}{\partial q_n^{i/k}} &= \sum_{N=0}^n \nu^{n-N} h_{n-N} \left(\frac{1}{i/k}, \frac{1}{i/k+1}, \dots, \frac{1}{i/k+N} \right) \delta_{kN+i}, \\ \frac{\partial}{\partial \bar{q}_n^{j/m}} &= \sum_{N=0}^n \bar{\nu}^{n-N} h_{n-N} \left(\frac{1}{j/m}, \frac{1}{j/m+1}, \dots, \frac{1}{j/m+N} \right) \bar{\delta}_{mN+j}. \end{aligned}$$

Following an argument of E. Getzler ([13], Proposition A.1) we show that the above formulas can be inverted. Namely,

Lemma 2.1. *The following formulas hold*

$$\begin{aligned} \delta_{kL+i} &= \sum_{n=0}^L \nu^{L-n} e_{L-n} \left(\frac{1}{i/k}, \frac{1}{i/k+1}, \dots, \frac{1}{i/k+L-1} \right) \frac{\partial}{\partial q_n^{i/k}}, \\ \bar{\delta}_{mL+j} &= \sum_{n=0}^L \bar{\nu}^{L-n} e_{L-n} \left(\frac{1}{j/m}, \frac{1}{j/m+1}, \dots, \frac{1}{j/m+L-1} \right) \frac{\partial}{\partial \bar{q}_n^{j/m}}. \end{aligned}$$

Proof. We prove the first identity. The argument for the second one is similar. We need to show that the following identity holds for any two integers $L \geq N$:

$$\begin{aligned} \sum_{n=N}^L \nu^{L-n} \nu^{n-N} e_{L-n} \left(\frac{1}{i/k}, \frac{1}{i/k+1}, \dots, \frac{1}{i/k+L-1} \right) \times \\ h_{n-N} \left(\frac{1}{i/k}, \frac{1}{i/k+1}, \dots, \frac{1}{i/k+N} \right) = \delta_N^L. \end{aligned}$$

If $L = N$ then the identity is obviously true. Assume that $L > N$. Then the LHS can be interpreted as the coefficient in front of ν^{L-N} in the product:

$$\prod_{a=0}^{L-1} (1 + \nu/(i/k + a)) \prod_{a=0}^N \frac{1}{(1 + \nu/(i/k + a))}.$$

However, with respect to ν , this is a polynomial of degree $L - N - 1$. \square

The proof of Theorem 1.2 amounts to changing the variables in the vertex operators Γ^\pm and $\bar{\Gamma}^\pm$. Let us begin with Γ^\pm and more precisely with the summands in (1.1) corresponding to $n > 0$ and $i = k - j$, $1 \leq j \leq k - 1$. The coefficient in front of λ^{-nk-j} transforms as follows:

$$\begin{aligned} & (\mathbf{1}_{j/k}(\nu + (j/k)z) \dots (\nu + (j/k + n - 1)z))^\wedge = \\ &= (n - 1 + j/k)! \sum_{l=0}^n (z^l \mathbf{1}_{j/k})^\wedge \nu^{n-l} e_{n-l} \left(\frac{1}{j/k}, \frac{1}{j/k + 1}, \dots, \frac{1}{j/k + n - 1} \right) = \\ &= (n - 1 + j/k)! \epsilon \delta_{kn+j} = \frac{1}{kn + j} \epsilon \partial_{y_{kn+j}}, \end{aligned}$$

We used that $(z^l \mathbf{1}_{j/k})^\wedge = \epsilon \partial / \partial q_l^{j/k}$ and the first identity in Lemma 2.1. Similarly, one can verify that the above answer is valid also for all pairs n, i such that either $n > 0$ and $i = k$, or $n = 0$ and $1 \leq i \leq k - 1$.

Let $y_{Nk+i} = \sum_{L \geq 0} a_{N,L} q_L^{i/k}$ be a linear change. Then by the chain rule: $\partial_{q_L^{i/k}} = \sum_{N \geq 0} a_{N,L} \partial_{y_{Nk+i}}$, i.e.,

$$\sum_{L \geq 0} (-w)^{-L-1} \partial_{q_L^{i/k}} = \sum_{N \geq 0} \left(\sum_{L \geq 0} a_{N,L} (-w)^{-L-1} \right) \partial_{y_{Nk+i}}.$$

On the other hand, since our linear change is defined by (1.4), we get

$$\sum_{L \geq 0} a_{N,L} (-w)^{-L-1} = \frac{g_{i/k}}{\prod_{l=0}^N (\nu - (l + i/k)w)}.$$

Note that with respect to the Poincaré pairing we have $\mathbf{1}_{(k-i)/k} = g_{i/k} \mathbf{1}^{i/k}$. The term corresponding to $n = -N - 1 < 0$ and i , $1 \leq i \leq k$, in the exponent of Γ^+ transforms as follows:

$$\left(\frac{g_{i/k}}{\prod_{l=0}^N (\nu - (l + i/k)z)} \mathbf{1}^{i/k} \right)^\wedge \lambda^{Nk+i} = \sum_{L \geq 0} a_{N,L} ((-z)^{-L-1} \mathbf{1}^{i/k})^\wedge \lambda^{Nk+i}.$$

On the other hand, according to our quantization rules, $((-z)^{-L-1} \mathbf{1}^{i/k})^\wedge = -\epsilon^{-1} q_L^{i/k}$. Thus the term corresponding to $n = -N - 1$ and i is $-\epsilon^{-1} y_{Nk+i} \lambda^{Nk+i}$.

Finally, the term corresponding to $n = 0$ and $i = k$ is $\widehat{\mathbf{1}}_{0/k}$. Thus, in the new coordinates, the vertex operators Γ^\pm are given by

$$\Gamma^\pm = \exp\left(\mp \sum_{n=1}^{\infty} \lambda^n y_n / \epsilon\right) \exp\left(\pm \sum_{n=1}^{\infty} \frac{\lambda^{-n}}{n} \epsilon \partial_{y_n}\right) e^{\pm \widehat{\mathbf{1}}_{0/k}}.$$

Similarly the other two vertex operators $\bar{\Gamma}^\pm$ are given by

$$\bar{\Gamma}^\pm = \exp\left(\mp \sum_{n=1}^{\infty} \lambda^n \bar{y}_n / \epsilon\right) \exp\left(\pm \sum_{n=1}^{\infty} \frac{\lambda^{-n}}{n} \epsilon \partial_{\bar{y}_n}\right) e^{\pm \widehat{\mathbf{1}}_{0/m}}.$$

Substitute these formulas in the HQE in Theorem 1.1 and note that by definition: $\mathcal{D}_n = Q^{n^2/2} e^{n(\widehat{\mathbf{1}}_{0/k} + \widehat{\mathbf{1}}_{0/m})} \mathcal{D}^{\text{Fr}}$. After a short simplification and up to rescaling y_n and \bar{y}_n by ϵ^{-n} we get the HQE of the 2-Toda hierarchy (see appendix B).

3. GROMOV-WITTEN THEORY OF $\mathbb{CP}_{k,m}^1$

3.1. The system of quantum differential equations. For some basics on orbifold Gromov-Witten theory we refer the reader to [9] and [1, 2]. We recall the vector space H which by definition coincides with the vector space of the equivariant cohomology algebra of the inertia orbifold $I\mathbb{CP}_{k,m}^1$. For each $\tau \in H$ the orbifold quantum cup product \bullet_τ is a commutative associative multiplication in H defined by the following genus-0 Gromov-Witten invariants:

$$(\mathbf{1}_\alpha \bullet_\tau \mathbf{1}_\beta, \mathbf{1}_\gamma) = \sum_{l,d \geq 0} \frac{Q^d}{l!} \int_{[\mathcal{M}_{0,l+3}(\mathbb{CP}_{k,m}^1; d)]^{\text{vir}}} \text{ev}^* (\mathbf{1}_\alpha \otimes \mathbf{1}_\beta \otimes \mathbf{1}_\gamma \otimes \tau^{\otimes l}),$$

where ev is the evaluation map at the $l + 3$ marked points. For brevity the RHS of the above equality will be denoted by the correlator $\langle \mathbf{1}_\alpha, \mathbf{1}_\beta, \mathbf{1}_\gamma \rangle_{0,3}(\tau)$. We use similar correlator notations for the other Gromov-Witten invariants as well.

It is a basic fact in quantum cohomology theory that the following system of ordinary differential equations is compatible:

$$z \partial_{\tau^\alpha} \Phi = \mathbf{1}_\alpha \bullet_\tau \Phi, \quad \alpha \in \mathbb{Z}_k \sqcup \mathbb{Z}_m,$$

where τ^α are the coordinates of H with respect to the basis $\mathbf{1}_\alpha$, and $\mathbf{1}_\alpha \bullet_\tau$ is the operator of quantum multiplication by $\mathbf{1}_\alpha$. This system is called the system of Quantum Differential Equations (QDE) of the orbifold $\mathbb{CP}_{k,m}^1$.

If the parameter z is close to ∞ then the following $\text{End}(H)$ -valued series $S = 1 + S_1 z^{-1} + \dots$ provides a fundamental solution to the system of QDE:

$$(S_\tau \phi_\alpha, \phi_\beta) = (\phi_\alpha, \phi_\beta) + \sum_{k=0}^{\infty} \langle \psi^k \phi_\alpha, \phi_\beta \rangle_{0,2}(\tau) z^{-k-1}.$$

Our first goal is to explicitly compute S_τ for $\tau \in H^2(\mathbb{CP}_{k,m}^1)$.

3.2. The J-function. The idea is to compute the so called J -function of $\mathbb{CP}_{k,m}^1$ defined by

$$J_{\mathbb{CP}_{k,m}^1}(\tau) = z \mathbf{1} + \tau + \sum_{k=0}^{\infty} \langle \mathbf{1}_\alpha \psi^k \rangle_{0,1}(\tau) \mathbf{1}^\alpha z^{-k-1}.$$

In this Section we calculate the restrictions to $H^2(\mathbb{CP}_{k,m}^1)$ of the J -function and its partial derivatives. Due to technical reasons we assume that k, m are co-prime. However we conjecture that the main results, Proposition 3.2, Corollaries 3.3 and 3.4 also hold in general. The general case will be addressed elsewhere using results of [7] concerning toric Fano stacks.

Suppose that k, m are co-prime. Then it is easy to see that $\mathbb{CP}_{k,m}^1$ is isomorphic to the weighted projective line $\mathbb{P}(k, m)$, which is defined to be the stack quotient $[(\mathbb{C}^2 - 0)/\mathbb{C}^*]$ under the following \mathbb{C}^* -action:

$$\lambda \cdot (z_0, z_1) = (\lambda^{-k} z_0, \lambda^{-m} z_1).$$

It is important to note that the identification of the isotropy groups at stacky points are different. For $\mathbb{P}(k, m)$, there is a natural map $\mathbb{P}(k, m) \rightarrow \mathbb{P}^1$ given by $[z_0; z_1] \mapsto [z_0^m; z_1^k]$. The neighborhood $\{z_0^m \neq 0\} \subset \mathbb{P}^1$ of $0 = [1; 0]$ has the coordinate z_1^k/z_0^m , and the stack structure over 0 is given by $z_1/z_0^{m/k} \mapsto z_1^k/z_0^m$ where \mathbb{Z}_k acts by multiplication by $\exp(\frac{-2\pi\sqrt{-1}m}{k})$, while in case of $\mathbb{CP}_{k,m}^1$, \mathbb{Z}_k acts by multiplication by $\exp(\frac{2\pi\sqrt{-1}}{k})$. The neighborhood $\{z_1^k \neq 0\} \subset \mathbb{P}^1$ of $\infty = [0; 1]$ has the coordinate z_0^m/z_1^k , and the stack structure over ∞ is given by $z_0/z_1^{k/m} \mapsto z_0^m/z_1^k$ where \mathbb{Z}_m acts by multiplication by $\exp(\frac{-2\pi\sqrt{-1}k}{m})$, while in case of $\mathbb{CP}_{k,m}^1$, \mathbb{Z}_m acts by multiplication by $\exp(\frac{2\pi\sqrt{-1}}{m})$.

The standard $T = (S^1)^2$ action on \mathbb{C}^2 descends to a $(S^1)^2$ action on $\mathbb{P}(k, m)$. This gives a $(S^1)^2$ -action on the line bundle $\mathcal{O}_{\mathbb{P}(k,m)}(1)$. Let λ_0/k and λ_1/m be the weights of this action at 0 and ∞ respectively.

Definition 3.1. For each real number r we denote by $\{r\} \in (0, 1]$ the unique real number s.t. $r - \{r\} \in \mathbb{Z}$. Note the range of $\{r\}$.

Proposition 3.2. The T -equivariant J -function of $X = \mathbb{CP}_{k,m}^1$ is given by the following formula

$$(3.1) \quad \begin{aligned} & z e^{\tau \nu_0/z} \sum_{d \in \mathbb{Z}_{\geq 0}} \frac{Q^{dm} e^{dm\tau}}{d! z^d \prod_{b=\{dm/k\}}^{dm/k} (\nu + bz)} \mathbf{1}_{-dm/k} + \\ & + z e^{\tau \nu_1/z} \sum_{d \in \mathbb{Z}_{\geq 0}} \frac{Q^{dk} e^{dk\tau}}{\prod_{b=\{dk/m\}}^{dk/m} (\bar{\nu} + bz) d! z^d} \mathbf{1}_{-dk/m}. \end{aligned}$$

where if $d = 0$ then both fractions are by definition 1 and in each product, b varies over all rational numbers which have the same fractional part as the corresponding upper (or lower) range of the product.

Proof. The calculation of [8], which is easily seen to apply to the T -equivariant setting, yields the following formula for the T -equivariant J -function for $\mathbb{P}(k, m)$:

$$(3.2) \quad \begin{aligned} & J_{\mathbb{P}(k,m)}(t) \\ &= z e^{Pt/z} \left(\mathbf{1}_0 \sum_{d \in \mathbb{Z}_{\geq 0}} \frac{Q^d e^{dt}}{\prod_{b_0=1}^{dk} (c_1^T(\mathcal{O}_{\mathbb{P}(k,m)}(k)) - \lambda_0 + b_0 z) \prod_{b_1=1}^{dm} (c_1^T(\mathcal{O}_{\mathbb{P}(k,m)}(m)) - \lambda_1 + b_1 z)} \right. \\ &+ \sum_{i=1}^{k-1} \mathbf{1}_{i/k} \sum_{d \in \mathbb{Z}_{\geq 0}} \frac{Q^{d+i/k} e^{(d+i/k)t}}{\prod_{b_0=1}^{kd+i} (c_1^T(\mathcal{O}_{\mathbb{P}(k,m)}(k)) - \lambda_0 + b_0 z) \prod_{b_1=\{im/k\}}^{dm+im/k} (c_1^T(\mathcal{O}_{\mathbb{P}(k,m)}(m)) - \lambda_1 + b_1 z)} \\ &+ \left. \sum_{j=1}^{m-1} \mathbf{1}_{j/m} \sum_{d \in \mathbb{Z}_{\geq 0}} \frac{Q^{d+j/m} e^{(d+j/m)t}}{\prod_{b_0=\{jk/m\}}^{dk+jk/m} (c_1^T(\mathcal{O}_{\mathbb{P}(k,m)}(k)) - \lambda_0 + b_0 z) \prod_{b_1=1}^{dm+j} (c_1^T(\mathcal{O}_{\mathbb{P}(k,m)}(m)) - \lambda_1 + b_1 z)} \right). \end{aligned}$$

Here $P = c_1^T(\mathcal{O}_{\mathbb{P}(k,m)}(1)) \in H^2(\mathbb{P}(k, m))$ and t is its coordinate. (3.1) follows from (3.2) by incorporating the following changes:

- In our notation, $\mathbf{1}_0 = \mathbf{1}_{0/k} + \mathbf{1}_{0/m}$.
- We want to measure degrees of curve classes using $\mathcal{O}^{\text{unif}}(1) = \mathcal{O}_{\mathbb{P}(k,m)}(km)$, where in [8] $\mathcal{O}_{\mathbb{P}(k,m)}(1)$ is used. As a consequence the degree of a curve class we want to use is km times theirs.
- We use the coordinate τ of the class $c_1^T(\mathcal{O}^{\text{unif}}(1))$ as the variable for J_X .
- We have the following equalities:

$$\begin{aligned} c_1^T(\mathcal{O}_{\mathbb{P}(k,m)}(k)) &= p/m, & c_1^T(\mathcal{O}_{\mathbb{P}(k,m)}(m)) &= p/k; \\ p \cdot \mathbf{1}_{i/k} &= p|_0 = m\lambda_0, & p \cdot \mathbf{1}_{j/m} &= p|_{\infty} = k\lambda_1; \\ \nu_0 &= m\lambda_0, & \nu_1 &= k\lambda_1. \end{aligned}$$

Using this we rewrite

$$\begin{aligned} \frac{1}{\prod_{1 \leq b \leq B} (c_1^T(\mathcal{O}_{\mathbb{P}(k,m)}(k)) - \lambda_0 + bz)} \mathbf{1}_{i/k} &= \frac{1}{B!z^B} \mathbf{1}_{i/k}, \\ \frac{1}{\prod_{1 \leq b \leq B} (c_1^T(\mathcal{O}_{\mathbb{P}(k,m)}(k)) - \lambda_0 + bz)} \mathbf{1}_{j/m} &= \frac{1}{\prod_{1 \leq b \leq B} (\bar{\nu} + bz)} \mathbf{1}_{j/m}, \\ \frac{1}{\prod_{1 \leq b \leq B} (c_1^T(\mathcal{O}_{\mathbb{P}(k,m)}(m)) - \lambda_1 + bz)} \mathbf{1}_{j/m} &= \frac{1}{B!z^B} \mathbf{1}_{j/m}, \\ \frac{1}{\prod_{1 \leq b \leq B} (c_1^T(\mathcal{O}_{\mathbb{P}(k,m)}(m)) - \lambda_1 + bz)} \mathbf{1}_{i/k} &= \frac{1}{\prod_{1 \leq b \leq B} (\nu + bz)} \mathbf{1}_{i/k}. \end{aligned}$$

- The difference in the identification of isotropy groups discussed above imposes the changes $\mathbf{1}_{i/k} \mapsto \mathbf{1}_{-im/k}$ and $\mathbf{1}_{j/m} \mapsto \mathbf{1}_{-kj/m}$.

□

A direct calculation gives the following

Corollary 3.3. *The J -function J_X satisfies the following differential equation:*

$$(3.3) \quad \prod_{i=0}^{k-1} \left(\frac{z}{m} \frac{\partial}{\partial \tau} - \frac{\nu_0}{m} - iz \right) \prod_{j=0}^{m-1} \left(\frac{z}{k} \frac{\partial}{\partial \tau} - \frac{\nu_1}{k} - jz \right) J_X = Q^{km} e^{km\tau} J_X$$

Corollary 3.4. *The restrictions of the partial derivatives of J_X to the small parameter space are given as follows:*

$$(3.4) \quad \begin{aligned} (kg_{i/k})^{-1} z \partial_{i/k} J_X &= ze^{\tau\nu_0/z} \sum_{d \in \mathbb{Z}_{\geq 0}} \frac{Q^{dm} e^{dm\tau}}{d! z^d} \frac{\prod_{b < \{(dm-i)/k\}} (\nu + bz)}{\prod_{b \leq (dm-i)/k} (\nu + bz)} \mathbf{1}_{(-dm+i)/k} \\ &+ ze^{\tau\nu_1/z} \sum_{d \in \mathbb{Z}_{\geq 0}} \frac{Q^{dk+i} e^{(dk+i)t}}{\prod_{b=\{(dk+i)/m\}}^{(dk+i)/m} (\bar{\nu} + bz) d! z^d} \mathbf{1}_{-(dk+i)/m}, \quad 1 \leq i \leq k; \end{aligned}$$

$$(3.5) \quad \begin{aligned} (mg_{j/m})^{-1} z \partial_{j/m} J_X &= ze^{\tau\nu_0/z} \sum_{d \in \mathbb{Z}_{\geq 0}} \frac{Q^{dm+j} e^{(dm+j)\tau}}{d! z^d \prod_{b=\{(dm+j)/k\}}^{(dm+j)/k} (\nu + bz)} \mathbf{1}_{-(dm+j)/k} \\ &+ ze^{\tau\nu_1/z} \sum_{d \in \mathbb{Z}_{\geq 0}} \frac{Q^{dk} e^{(dk)t}}{d! z^d} \frac{\prod_{b < \{(dk-j)/m\}} (\bar{\nu} + bz)}{\prod_{b \leq (dk-j)/m} (\bar{\nu} + bz)} \mathbf{1}_{(-dk+j)/m}, \quad 1 \leq j \leq m, \end{aligned}$$

where the notations and the conventions are the same as above.

The idea of the proof, borrowed from Section 5 of [8], is to express the partial derivatives of the J -function as linear combinations of derivatives along $H^2(\mathbb{CP}_{k,m}^1)$. This is possible only when k and m are co-prime. The computation is straightforward but a bit technical. It will be given in Appendix A.

3.3. Equivariant quantum cohomology of $\mathbb{CP}_{k,m}^1$. Put $N := k + m$. Let k and m be co-prime numbers. Then as it was explained above, $\mathbb{CP}_{k,m}^1$ is isomorphic as an orbifold to the weighted projective line. We recall [8] Corollary 1.2. The proof of this corollary (see [8] section 5) generalizes to equivariant settings and we get the following description of the equivariant quantum cup product of $\mathbb{CP}_{k,m}^1$ at a point $\tau = t_N p$, $t_N \in \mathbb{C}$. The map

$$\begin{aligned} \mathbf{1}_{i/k} &\mapsto \phi_{i/k} := x^i, \quad 1 \leq i \leq k-1, \quad \mathbf{1}_{0/k} \mapsto \phi_{0/k} := kx^k/(\nu_0 - \nu_1), \\ \mathbf{1}_{j/m} &\mapsto \phi_{j/m} := y^j, \quad 1 \leq j \leq m-1, \quad \mathbf{1}_{0/m} \mapsto \phi_{0/m} := my^m/(\nu_1 - \nu_0), \end{aligned}$$

identifies the algebra (H, \bullet_τ) , with the algebra

$$\mathbb{C}[x, x^{-1}] / \langle \partial_x f \rangle,$$

where $f = x^k + (Qe^{t_N}/x)^m + \nu_1 \log x + \nu_0 \log(Qe^{t_N}/x)$.

The description of small orbifold quantum cohomology of $\mathbb{CP}_{k,m}^1$ is in fact valid without assuming that k, m are co-prime. This may be seen as follows. Put $x_i = \mathbf{1}_{i/k}, y_j = \mathbf{1}_{j/m}$. Then in equivariant orbifold cohomology it is easy to see that

$$\begin{aligned} x_i \cdot y_j &= 0, \quad i, j \neq 0; \\ x_{i_1} \cdot x_{i_2} &= x_{i_1+i_2}, \quad i_1 + i_2 \leq k-1; \\ y_{j_1} \cdot y_{j_2} &= y_{j_1+j_2}, \quad j_1 + j_2 \leq m-1. \end{aligned}$$

Also,

$$\begin{aligned} x_{k-1} \cdot x &= \langle x_{k-1}, x, 1 \rangle_{0,3,0} P.D.(1) + \langle x_{k-1}, x, p \rangle_{0,3,0} P.D.(p) \\ &= p/k + \nu_0/k \end{aligned}$$

since $\langle x_{k-1}, x, 1 \rangle_{0,3,0} = 1/k, \langle x_{k-1}, x, p \rangle_{0,3,0} = \int_{B\mathbb{Z}_k} p = \nu_0/k$, and $P.D.(1) = p, P.D.(p) = 1$. Similarly we have $y_{m-1} \cdot y = p/m + \nu_1/m$. So the equivariant orbifold cohomology algebra can be identified with

$$\mathbb{C}[\nu_0, \nu_1][x, y] / (kx^k - \nu_0 = my^m - \nu_1, xy = 0),$$

where $x := x_1, y := y_1$.

To calculate the small equivariant orbifold quantum cohomology we only need to find the correct deformations of the two relations $kx^k - \nu_0 = my^m - \nu_1, xy = 0$. We will use the fact that the small equivariant orbifold quantum cohomology algebra is *graded* as a \mathbb{C} -algebra, with $\deg x = 1/k, \deg y = 1/m, \deg \nu_0 = \deg \nu_1 = 1, \deg q = 1/k + 1/m$. By degree reason it is easy to see that the relation $xy = 0$ is deformed to $xy = q$. The relation $kx^k - \nu_0 = my^m - \nu_1$ remains undeformed. This can be seen in the same way as its non-equivariant counterpart treated in [20], Section 4.3. Thus the small equivariant orbifold quantum cohomology of $\mathbb{CP}_{k,m}^1$ is isomorphic to

$$\mathbb{C}[[q]][\nu_0, \nu_1][x, y] / (kx^k - \nu_0 = my^m - \nu_1, xy = q).$$

Relationship between small quantum cohomology and big quantum cohomology restricted to H^2 imposes the change of variable $q = Qe^{t_N}$. This yields the description above.

We conjecture that the full equivariant quantum cohomology can be described in a similar way. Namely, let M be the family of functions on the complex circle \mathbb{C}^* of the type:

$$f_t = x^k + \sum_{i=1}^k t_i x^{k-i} + \sum_{j=1}^{m-1} t_{k+j} \left(\frac{Qe^{t_N}}{x} \right)^j + \left(\frac{Qe^{t_N}}{x} \right)^m + \nu_1 \log x + \nu_0 \log \left(\frac{Qe^{t_N}}{x} \right).$$

Each tangent space of M is equipped with an algebra structure via the map:

$$T_t M \cong \mathbb{C}[x, x^{-1}] / \langle \partial_x f_t \rangle, \quad \partial / \partial t_i \mapsto [\partial f_t / \partial t_i], \quad 1 \leq i \leq N.$$

Let $\omega := dx/x$ be the standard volume form on \mathbb{C}^* . We equip each tangent space $T_t M$ with a residue metric:

$$([\phi_1], [\phi_2])_t = -(\text{res}_{x=0} + \text{res}_{x=\infty}) \frac{\phi_1 \omega \phi_2 \omega}{df_t}.$$

We claim that this is a flat metric on M and we prove it by constructing explicitly a coordinate system on M such that the metric is constant. If x is close to ∞ then the equation

$$(3.6) \quad f_t(x) = \lambda^k + \nu_1 \log \lambda + \nu_0 \log(Q/\lambda)$$

admits a unique solution of the type $x = \lambda + a_0(t) + a_1(t)\lambda^{-1} + \dots$, i.e., the equation determines a coordinate change near $x = \infty$ and we have the following expansion

$$\log x = \log \lambda - \frac{1}{k} \left(\sum_{i=1}^k \tau^{i/k} \lambda^{-i} \right) + O(\lambda^{-k-1}),$$

where $\tau^{i/k}$ are polynomials in $t = (t_1, t_2, \dots, t_N)$. More precisely, by using

$$(3.7) \quad (i/k) \tau^{i/k} = -\text{res}_{x=\infty} \lambda^i \omega, \quad 1 \leq i \leq k,$$

we get

$$\begin{aligned} \tau^{1/k} &= t_1, \\ \tau^{i/k} &= t_i + f_{i/k}(t_1, \dots, t_{i-1}), \quad 2 \leq i \leq k-1, \\ \tau^{0/k} &= t_k + \nu_0 t_N, \end{aligned}$$

where $f_{i/k}$ are polynomials in t_1, \dots, t_{i-1} of degrees ≥ 2 . They can be computed explicitly by taking the coefficient in front of x^{-i} in the following Laurent polynomial:

$$\frac{1}{i/k} \sum_{n=2}^i \binom{i/k}{n} \left(\frac{t_1}{x} + \dots + \frac{t_{i-1}}{x^{i-1}} \right)^n.$$

This formula is obtained from formula (3.7) by truncating the terms in the change (3.6) that do not contribute to the residue in (3.7).

The rest of the flat coordinates can be constructed in a similar way. Let $y = Qe^{t_N}/x$ be another coordinate on the complex circle. Then each $f_t \in M$ assumes the form:

$$y^m + \sum_{j=1}^m t_{k+m-j} y^{m-j} + \sum_{i=1}^{k-1} t_{k-i} \left(\frac{Qe^{t_N}}{y} \right)^i + \left(\frac{Qe^{t_N}}{y} \right)^k + \nu_0 \log y + \nu_1 \log \left(\frac{Qe^{t_N}}{y} \right).$$

If y is close to ∞ then the equation

$$(3.8) \quad f_t(y) = \lambda^m + \nu_0 \log \lambda + \nu_1 \log(Q/\lambda)$$

determines a new coordinate near $y = \infty$ and we have the following expansion:

$$\log y = \log \lambda - \frac{1}{m} \left(\sum_{j=1}^m \tau^{j/m} \lambda^{-j} \right) + O(\lambda^{-m-1}),$$

where $\tau^{j/m}$ are polynomials in $t = (t_1, t_2, \dots, t_N)$. The same arguments as above yield the following:

$$\begin{aligned} \tau^{1/m} &= t_{k+m-1}, \\ \tau^{j/m} &= t_{k+m-j} + f_{j/m}(t_{k+m-1}, \dots, t_{k+m-(j-1)}), \quad 2 \leq j \leq m-1, \\ \tau^{0/m} &= t_k + \nu_1 t_N, \end{aligned}$$

where $f_{j/m}$ are polynomials of degrees at least 2 and can be computed explicitly by taking the coefficient in front of y^{-j} of the following Laurent polynomial:

$$\frac{1}{j/m} \sum_{n=2}^j \binom{j/m}{n} \left(\frac{t_{k+m-1}}{y} + \dots + \frac{t_{k+m-(j-1)}}{y^{j-1}} \right)^n.$$

Lemma 3.5. *In the coordinate system $\{\tau^\alpha\}_{\alpha \in \mathbb{Z}_k \sqcup \mathbb{Z}_m}$, the residue pairing coincides with the Poincaré pairing. More precisely:*

$$(\partial/\partial\tau^\alpha, \partial/\partial\tau^\beta) = (\mathbf{1}_\alpha, \mathbf{1}_\beta).$$

Proof. We prove the equality only when $\alpha = i/k, \beta = i'/k, 1 \leq i, i' \leq k$. The other cases may be treated in a similar way. Let us compute the residue at $x = \infty$ in the residue pairing. We change from x to the coordinate λ defined by equation (3.6). Differentiation by parts yields $\partial_{\tau^\alpha} f_t + f'_t (\partial_{\tau^\alpha} x) = 0$. Therefore

$$\partial_{\tau^\alpha} f_t \omega = -(\partial_{\tau^\alpha} \log x) df_t = k^{-1} (\lambda^{-i} + O(\lambda^{-k-1})) (k\lambda^k + \nu_1 - \nu_0) \frac{d\lambda}{\lambda}.$$

Now, the $(-\text{res}_{x=\infty})$ -term in the residue pairing of $(\partial/\partial\tau^\alpha, \partial/\partial\tau^\beta)$ equals to

$$-\text{res}_{\lambda=\infty} k^{-2} (k\lambda^{k-i} + (\nu_1 - \nu_0)\lambda^{-i} + O(\lambda^{-k-1})) (\lambda^{-i'} + O(\lambda^{-k-1})) \frac{d\lambda}{\lambda}.$$

The last residue equals $1/k$ if $i + i' = k$ and 0 otherwise. To compute the $(-\text{res}_{x=0})$ -term in the residue pairing, we switch to the coordinate $y = Qe^{t_N}/x$ and then, in a neighborhood of $y = \infty$, we change to the coordinate λ defined by equation (3.8). An extra caution is required here since the 1-form in the residue involves partial derivatives in τ^α and τ^β and the coordinate change depends on t . Put $\tilde{f}_t = f_t(Qe^{t_N}/y)$. Then differentiation by parts yields

$$(\partial_{\tau^\alpha} f_t) \omega = \left(-\frac{\partial \tilde{f}_t}{\partial \tau^\alpha} \frac{dy}{y} + \frac{\partial t_N}{\partial \tau^\alpha} d\tilde{f}_t \right) = \left(\partial_{\tau^\alpha} \log y + \frac{\partial t_N}{\partial \tau^\alpha} \right) d\tilde{f}_t.$$

In the last formula if we change from y to λ then we get

$$(\partial_{\tau^\alpha} f_t) \omega = \left(\frac{\partial t_N}{\partial \tau^\alpha} + O(\lambda^{-m-1}) \right) (m\lambda^m + (\nu_0 - \nu_1)) \frac{d\lambda}{\lambda}.$$

From here we find that the $(-\text{res}_{x=0})$ -term in the residue pairing of $(\partial/\partial\tau^\alpha, \partial/\partial\tau^\beta)$ equals to

$$-\text{res}_{\lambda=\infty} \left(\frac{\partial t_N}{\partial \tau^\alpha} \frac{\partial t_N}{\partial \tau^\alpha} + O(\lambda^{-m-1}) \right) (m\lambda^m + (\nu_0 - \nu_1)) \frac{d\lambda}{\lambda}.$$

On the other hand $t_N = (\tau^{0/k} - \tau^{0/m})/(\nu_0 - \nu_1)$. Therefore, the above residue is 0 unless $\alpha = \beta = 0/k$, in which case it equals $\frac{1}{\nu_0 - \nu_1}$. \square

We trivialize the tangent bundle $TM \cong M \times H$ via the flat coordinates, i.e., $\partial/\partial\tau^\alpha \mapsto \mathbf{1}_\alpha$. Let us denote by \bullet'_τ the multiplication in the tangent space $T_\tau M \cong H$.

Conjecture 3.6. *The equivariant cup product \bullet_τ coincides with \bullet'_τ .*

This may be interpreted as saying that f_t is the equivariant mirror of $\mathbb{CP}_{k,m}^1$.

As discussed above, we know that the conjecture holds for $\tau = t_N p$.

Suppose that k and m are co-prime, then the equivariant orbifold quantum cohomology of $\mathbb{CP}_{k,m}^1$ is multiplicatively generated in degree 2. Thus the conjecture follows from the reconstruction result of abstract quantum D-module ([17], Theorem 4.9).

For general k, m , we know from our previous article [20] that in the non-equivariant limit $\nu_0 = \nu_1 = 0$ the conjecture also holds. We expect that there should be a reconstruction-type theorem in equivariant quantum cohomology that implies the conjecture from the facts that we already know. However, reconstruction of non-conformal (i.e. the structure constants of the cup product are not homogeneous functions) Frobenius manifolds, is a topic in Gromov–Witten theory not explored yet. On the other hand it is not entirely true that the homogeneity is lost, because we can assign degree 2 (or 1 if we work with complex degrees) to each of the characters ν_0 and ν_1 and then the structure constants will be homogeneous. However, we could not find a way to use this homogeneity property.

We would like to remark that the Frobenius manifold M in this section is a slight generalization of the Frobenius structure on the space of orbits of the extended affine Weyl group of type A , introduced by B. Dubrovin in [10]. In particular our arguments are parallel to the ones in [10]. Apparently, a similar Frobenius manifold was introduced by J. Ferguson and I. Strachan (see [12]) in their study of logarithmic deformations of the dispersionless KP-hierarchy.

3.4. Oscillating integrals. Let $\tau \in M \cong H$ be such that f_τ is a Morse function. Denote by $\xi_i \in \mathbb{C}^*$, $i = 1, 2, \dots, k+m$ the critical points of f_τ . For

each i we choose a semi-infinite homology cycle B_i in

$$\lim_{M \rightarrow \infty} H_1(Y_\tau, \{\operatorname{Re}(f_\tau/z) < -M\}; \mathbb{Z}) \cong \mathbb{Z}^{k+m}.$$

as follows. Pick a Riemannian metric on \mathbb{C}^* and let B_i be the union of the gradient trajectories of $\operatorname{Re}(f_\tau/z)$ which flow into the critical point ξ_i . We remark that the function f_τ is multivalued, however its gradient is a smooth vector field on \mathbb{C}^* and so the above definition of B_i makes sense. Let $\mathfrak{J} : \mathbb{C}^{k+m} \rightarrow H$ be the linear map defined by

$$(3.9) \quad (\mathfrak{J}e_i, \mathbf{1}_\alpha) := (-2\pi z)^{-1/2} \int_{B_i} e^{f_\tau/z} \phi_\alpha \omega,$$

where e_i , $i = 1, 2, \dots, k+m$ is the standard basis of \mathbb{C}^{k+m} , the index $\alpha \in \mathbb{Z}_k \sqcup \mathbb{Z}_m$, and we also fixed a choice of a branch of f_τ in a tubular neighborhood of the cycle B_i .

Using the method of stationary phase asymptotics (e.g. see [3]) we get that the map \mathfrak{J} admits the following asymptotic:

$$(3.10) \quad \mathfrak{J} \sim \Psi(1 + R_1 z + R_2 z^2 + \dots) e^{U/z}, \text{ as } z \rightarrow 0,$$

where R_1, R_2, \dots and $U = \operatorname{diag}(u_1, \dots, u_{k+m})$ ($u_i = f_\tau(\xi_i)$ are the critical values of f_τ) are linear operators in \mathbb{C}^{k+m} , and $\Psi : \mathbb{C}^{k+m} \rightarrow H$ is a linear isomorphism (independent of z). Under the isomorphism Ψ , the product \bullet_τ and the residue pairing are transformed respectively into

$$e_i \bullet_\tau e_j = \Delta_i^{1/2} \delta_{i,j} e_j, \quad (e_i, e_j) = \delta_{i,j},$$

where $\delta_{i,j}$ is the Kronecker symbol and Δ_i is the Hessian of f_τ at the critical point ξ_i with respect to the volume form ω_τ , i.e., choose a *unimodular* coordinate t in a neighborhood of ξ_i so that $\omega = dt$ and then $\Delta_i = \partial_t^2 f_\tau(\xi_i)$. We will write $R = 1 + R_1 z + R_2 z^2 + \dots$

We are ready to define the function \mathcal{D}^{Fr} . However before we do this let us list two more facts which are not needed in the sequel but will be important for proving that \mathcal{D}^{Fr} coincides with the total descendent potential of $\mathbb{CP}_{k,m}^1$.

Theorem 3.7. *The map \mathfrak{J} satisfies the following differential equations:*

$$(3.11) \quad z \partial_{\tau^\alpha} \mathfrak{J} = (\phi_\alpha \bullet'_\tau) \mathfrak{J}, \quad \alpha \in \mathbb{Z}_k \sqcup \mathbb{Z}_m.$$

The proof of this theorem will be omitted because it is the same as the proof of Lemma 3.1 in [20].

Assume that $\tau = t_N p$ and that the critical points ξ_i of f_τ are numbered in such a way that

$$\xi_i = \nu^{1/k} + \dots, \quad 1 \leq i \leq k \text{ and } \xi_{k+j} = Q e^{\tau} \bar{\nu}^{-1/m} + \dots, \quad 1 \leq j \leq m,$$

where the two groups of expansions are obtained by solving $f'_\tau(x) = 0$ respectively in a neighborhood of $x = \infty$ and $x = 0$, the dots stand for higher order

terms in Q , and the index i (resp. j) corresponds to a choice of k -th root of ν (resp. m -th root of $\bar{\nu}$). Put

$$\begin{aligned} g_{\alpha i} &:= g_{\alpha} \nu^{j/k-1/2}, \quad \alpha = j/k \in \mathbb{Z}_k, \quad 1 \leq j \leq k, \quad 1 \leq i \leq k, \\ g_{\alpha i} &:= g_{\alpha} \bar{\nu}^{j/m-1/2}, \quad \alpha = j/m \in \mathbb{Z}_m, \quad 1 \leq j \leq m, \quad k+1 \leq i \leq k+m, \end{aligned}$$

Lemma 3.8. *The asymptotical solution admits a classical limit $Q = 0$ which is characterized as follows: $(\Psi Re_i, \mathbf{1}_{\alpha})$, turns into either*

$$(3.12) \quad g_{\alpha i} \exp \left(\sum_{n=2}^{\infty} \frac{B_n(1-j/k)}{n(n-1)} (-\nu)^{-n+1} z^{n-1} \right)$$

if $\alpha = j/k$, $1 \leq j \leq k$, or

$$(3.13) \quad g_{\alpha i} \exp \left(\sum_{n=2}^{\infty} \frac{B_n(1-j/m)}{n(n-1)} (-\bar{\nu})^{-n+1} z^{n-1} \right),$$

if $\alpha = j/m \in \mathbb{Z}_m$, $1 \leq j \leq m$, where $B_n(x)$ are the Bernoulli polynomials:

$$\frac{e^{xt}t}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.$$

Proof. It is enough to verify the first asymptotic, because for the second one we can employ the symmetry: switch ν_0 with ν_1 and k with m . We have to compute the asymptotic of (3.9) up to higher order terms in Q . Therefore we can use $x^k + (\nu_1 - \nu_0) \log x$ instead of f_{τ} and also we can assume that e_i corresponds to the critical point ξ_i , $1 \leq i \leq k$. Let us make the substitution $t = x^k$. Then the integral (3.9), modulo higher order terms in Q , turns into

$$(3.14) \quad k^{-1} (-2\pi z)^{-1/2} \int_B e^{(t-\nu \log t)z^{-1}} \phi_{\alpha}(t^{1/k}) t^{-1} dt,$$

where the cycle B is constructed via Morse theory for $\operatorname{Re} (t - \nu \log t)/z$ (see the construction of B_i in (3.9)).

More generally, we will compute explicitly the asymptotic as $z \rightarrow 0$ of

$$(3.15) \quad I(\nu, z, s) = \int_B e^{(t-\nu \log t)z^{-1}} t^{s-1} dt,$$

where $s > 0$ is any real number. Using the method of stationary phase asymptotic (see [3]) we get that (3.15) admits an asymptotic as $z \rightarrow 0$ of the following type:

$$e^{(\nu - \nu \log \nu)/z} \nu^{s-1} (-2\pi \nu z)^{1/2} e^{\sum_{n=2}^{\infty} A_n(s) (-z/\nu)^{n-1}}.$$

In order to verify that the sum in the exponent depends on z/ν note that the integral (3.15) satisfies the differential equation $(z\partial_z + \nu\partial_{\nu})I = ((-\nu/z) + s)I$.

Furthermore, one checks that I satisfies the differential equation $(z\partial_\nu + \partial_s)I = 0$ which imposes the following recursive relations on the polynomials A_n :

$$(3.16) \quad A'_2(s) = s - \frac{1}{2}, \quad A'_{n+1}(s) = -A_n(s).$$

On the other hand when $s = 1$ the asymptotic of (3.15) is easily expressed in terms of the asymptotic of the Gamma function:

$$(-z)^{-\nu/z+1} \Gamma\left(-\frac{\nu}{z} + 1\right) \sim e^{(\nu-\nu \log \nu)/z} (-2\pi\nu z)^{1/2} e^{\sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n-1)} (-z/\nu)^{2n-1}},$$

where $B_n = B_n(0)$ are the Bernoulli numbers and the asymptotic of the Gamma function is well known (e.g. see [4]). Thus the coefficient A_n satisfy the following initial condition $A_n(1) = B_n/(n(n-1))$ (note that for $n \geq 2$ the odd Bernoulli numbers vanish), which together with (3.16) uniquely determines A_n . Using that the Bernoulli polynomials satisfy the identity: $B'_n(x) = nB_{n-1}(x)$, it is easy to verify that $A_n(s) = B_n(1-s)/(n(n-1))$. \square

Remark 3.9. *Lemma 3.8 implies Givental's R-conjecture for $\mathbb{CP}_{k,m}^1$.*

3.5. The symplectic loop space formalism. Let $\mathcal{H} := H((z^{-1}))$ be the space of formal Laurent series in z^{-1} with coefficients in H . We equip \mathcal{H} with the symplectic form:

$$\Omega(\mathbf{f}(z), \mathbf{g}(z)) := \text{res}_{z=0} (\mathbf{f}(-z), \mathbf{g}(z)) dz.$$

Let $\{\mathbf{1}^\alpha\}_{\alpha \in \mathbb{Z}_k \sqcup \mathbb{Z}_m}$ be a basis of H dual to $\{\mathbf{1}_\alpha\}$ with respect to the Poincaré pairing. Then the functions $p_{n,\alpha} = \Omega(\mathbf{1}_\alpha z^n)$ and $q_n^\alpha = \Omega(\mathbf{1}^\alpha (-z)^{-n-1})$, where $n \geq 0$ and $\alpha \in \mathbb{Z}_k \sqcup \mathbb{Z}_m$ form a Darboux coordinate system on \mathcal{H} . We quantize functions on \mathcal{H} via the Weyl's quantization rules: the coordinate functions $p_{n,\alpha}$ and q_n^α are represented respectively by the differential operator $\widehat{p}_{n,\alpha} = \epsilon \partial / \partial q_n^\alpha$ and the multiplication operator $\widehat{q}_n^\alpha = \epsilon^{-1} q_n^\alpha$, and we demand normal ordering, i.e., always put the differentiation before the multiplication operators.

If A is an infinitesimal symplectic transformation of \mathcal{H} then the map $\mathbf{f} \mapsto A\mathbf{f}$ determines a linear *Hamiltonian* vector field. It is straightforward to verify that the corresponding Hamiltonian coincides with the quadratic function $h_A := -\frac{1}{2}\Omega(A\mathbf{f}, \mathbf{f})$. By definition $\widehat{A} := \widehat{h}_A$. If M is a symplectic transformation of \mathcal{H} such that $A := \log M$ exist then we define $\widehat{M} := e^{\widehat{A}}$.

From now on we will consider only $\tau = t_N p$. Put

$$(3.17) \quad \mathcal{D}^{\text{Fr}} = C(\tau) \widehat{S}_\tau^{-1} (\Psi R e^{U/z})^\wedge \prod_{i=1}^{k+m} \mathcal{D}_{\text{pt}}(\mathbf{q}^i),$$

where the vector space H is identified with the standard vector space \mathbb{C}^{k+m} via Ψ and \mathbf{q}^i are the coordinates of $\mathbf{q} \in H[z]$ with respect to the standard

basis, i.e., $\sum \mathbf{q}^i(z)e_i = \Psi^{-1}\mathbf{q}(z)$, and \mathcal{D}_{pt} is the total descendent potential of a point:

$$\mathcal{D}_{\text{pt}}(\mathbf{t}) = \exp \left(\sum_{n,g} \epsilon^{2g-2} \frac{1}{n!} \int_{\overline{M}_{g,n}} \prod_{j=1}^n (\mathbf{t}(\psi_j) + \psi_j) \right),$$

where $\mathbf{t}(z) = t_0 + t_1 z + \dots \in \mathbb{C}[z]$. The factor C in (3.17) is a complex-valued function on H such that it makes the RHS independent of τ . For all further purposes $C(\tau)$ is irrelevant and it will be ignored.

4. VERTEX OPERATORS AND THE EQUIVARIANT MIRROR MODEL OF $\mathbb{CP}_{k,m}^1$

4.1. Introduction. Given a vector $\mathbf{f} \in \mathcal{H}$, the corresponding linear function $\Omega(\cdot, \mathbf{f})$ is a linear combination of $p_{n,\alpha}$ and q_n^α and $\widehat{\mathbf{f}}$ is defined by the above rules. Expressions like $e^{\mathbf{f}}$, $\mathbf{f} \in \mathcal{H}$ are quantized by first decomposing $\mathbf{f} = \mathbf{f}_- + \mathbf{f}_+$, where \mathbf{f}_+ (respectively \mathbf{f}_-) is the projection of \mathbf{f} on $\mathcal{H}_+ := H[z]$ (respectively $\mathcal{H}_- := z^{-1}H[[z^{-1}]]$), and then setting $(e^{\mathbf{f}})^\wedge = e^{\widehat{\mathbf{f}}_-} e^{\widehat{\mathbf{f}}_+}$. Note that the vertex operators in the introduction are quantized exactly in this way.

The proof of Theorem 1.1 amounts to conjugating the vertex operators Γ^\pm and $\overline{\Gamma}^\pm$ by the symplectic transformation \widehat{S}_τ and then by $\Psi Re^{U/z}$. For the first conjugation we use the following formula ([14], formula (17)):

$$(4.1) \quad \widehat{S}_\tau e^{\widehat{\mathbf{f}}} \widehat{S}_\tau^{-1} = e^{W(\mathbf{f}_+, \mathbf{f}_+)/2} e^{(S_\tau \mathbf{f})^\wedge},$$

where $\mathbf{f} \in \mathcal{H}$ and $+$ means truncating the terms corresponding to the negative powers of z and the quadratic form $W(\mathbf{f}_+, \mathbf{f}_+) = \sum (W_{nl} \mathbf{f}_l, \mathbf{f}_n)$ is defined by

$$(4.2) \quad W_{nl} w^{-n} z^{-l} = \frac{S_\tau^*(w) S_\tau(z) - 1}{w^{-1} + z^{-1}}.$$

Therefore, our next goal is to compute $S_\tau \mathbf{f}^\pm$ and $S_\tau \overline{\mathbf{f}}^\pm$. Before doing so we explain a very important property of our vertex operators. The content of the next section is the key to the proof of Theorem 1.1.

4.2. Changing the coordinate λ . Let us denote by \mathcal{O} the space of formal Laurent series in λ^{-1} and by $\mathcal{O}[[z^{\pm 1}]]$ the space of formal series:

$$(4.3) \quad \mathbf{f}(\lambda, z) = \sum_{n \in \mathbb{Z}} \mathcal{I}^{(n)}(\lambda) (-z)^n, \quad \text{such that} \quad \lim_{n \rightarrow \infty} \mathcal{I}^{(n)}(\lambda) = 0,$$

where the limit is understood in the λ -adic sense, i.e., for each $N > 0$ there exist $d \in \mathbb{Z}$ such that $\mathcal{I}^{(n)} \in \lambda^{-N} \mathbb{C}[[\lambda^{-1}]]$ for all $n \geq d$.

Furthermore, we fix an element $\phi \in \mathcal{O}$ such that both $\text{res}_{\lambda=\infty} \phi$ and the polynomial part $p \in \mathbb{C}[\lambda]$ of ϕ are non-zero and we introduce the following first order differential operator:

$$(4.4) \quad D = -z p^{-1} \partial_\lambda - p^{-1} \phi_-,$$

where $\phi_- := \phi - p$.

Let $\mathbf{g} \in \mathcal{O}[[z^{\pm 1}]]$ be a Laurent series in λ^{-1} and z^{-1} :

$$(4.5) \quad \mathbf{g} = \sum_{\substack{a \geq A \\ r \geq R}} g_{a,r} \lambda^{-a} z^{-r}.$$

We will prove that the operator D is a linear isomorphism in $\mathcal{O}[[z^{\pm 1}]]$ and that the infinite sum

$$(4.6) \quad \mathbf{f} = \sum_{n \in \mathbb{Z}} \mathcal{I}^{(n)}(\lambda) (-z)^n := \sum_{n \in \mathbb{Z}} D^n \mathbf{g},$$

is a well defined element of $\mathcal{O}[[z^{\pm 1}]]$. The main result in this subsection is the following transformation law for \mathbf{f} .

Proposition 4.1. *If $x = \lambda + a_0 + a_1 \lambda^{-1} + \dots$ is another formal coordinate near $\lambda = \infty$. Then*

$$\mathbf{f}(x) = \mathbf{f}(\lambda) \exp \left(z^{-1} \int_x^\lambda \phi(t) dt \right).$$

Proof. Note that $D\mathbf{f} = \mathbf{f}$, i.e.,

$$\partial_\lambda \mathcal{I}^{(n)}(\lambda) = \phi(\lambda) \mathcal{I}^{(n+1)}(\lambda).$$

Thus the same proof as in [19], Lemma 3.2, applies. \square

We will show that for each pair of positive integers M and N there exists $d \in \mathbb{Z}$ such that

$$(4.7) \quad D^{-M'} \mathbf{g} \in z^{-M} \mathcal{O}[[z^{-1}]] \quad \text{and} \quad D^{N'} \mathbf{g} \in \lambda^{-N} \mathbb{C}[[\lambda^{-1}, z^{\pm 1}]]$$

for all $M' > d$ and $N' > d$. This would imply that the infinite sum (4.6) is convergent in an appropriate z, λ -adic sense to some element in $\mathcal{O}[[z^{\pm 1}]]$.

We pass to a new variable $\xi = \int p(\lambda) d\lambda$. If $k - 1$ is the degree of the polynomial p then, after inverting the change, we see that $\mathcal{O} \cong \mathbb{C}((\xi^{-1/k}))$. Also the differential operator D takes the form

$$D_\xi = -z \partial_\xi + \nu / \xi + \sum_{i \geq 1} a_i \xi^{-1-i/k},$$

where a_i are some constants and $\nu \neq 0$.

Lemma 4.2. *The operator D_ξ is a linear isomorphism in $\mathcal{O}[[z^{\pm 1}]]$ and its inverse has the following property:*

$$D_\xi^{-1} \xi^\alpha \in \begin{cases} z^{-1} \mathcal{O}[[z^{-1}]] & \text{if } \alpha \neq -1, \\ \nu^{-1} + z^{-1} \mathcal{O}[[z^{-1}]] & \text{otherwise,} \end{cases}$$

where $\alpha \in (1/k)\mathbb{Z}$.

Proof. We will construct the inverse of D_ξ . The equation $D_\xi \mathbf{f} = \xi^\alpha$ has a unique solution of the following form

$$\mathbf{f}^\alpha = \frac{1}{-z(\alpha+1)+\nu} \xi^{\alpha+1} + \sum_{j \geq 1} f_j^\alpha \xi^{\alpha+1-j/k},$$

where $f_j^\alpha \in z^{-1}\mathbb{C}[[z^{-1}]]$. We define $D_\xi^{-1}\xi^\alpha := \mathbf{f}^\alpha$ and one checks that if D_ξ^{-1} is extended by linearity, then $D_\xi^{-1}\mathbf{f} \in \mathcal{O}[[z^{\pm 1}]]$ for all $\mathbf{f} \in \mathcal{O}[[z^{\pm 1}]]$. The lemma follows. \square

Assume that \mathbf{g} is a series of the type (4.5), i.e., the powers of z and λ are bounded from above. According to Lemma 4.2, the operator D^{-2} will decrease the highest degree of z at least by 1. On the other hand note that D decreases the highest degree of λ at least by k . Thus (4.7) holds.

4.3. The symplectic action on \mathbf{f}^\pm . Let D_x be the differential operator (4.4) corresponding to $\phi = \partial_x f_\tau$, i.e.,

$$(4.8) \quad D_x = -z \frac{1}{k} x^{1-k} \partial_x + \frac{1}{k} (\nu_0 - \nu_1) x^{-k} + \frac{m}{k} (Qe^\tau)^m x^{-k-m}.$$

We define a vector in the symplectic loop space \mathcal{H}

$$\mathbf{f}_\tau^\pm = \sum_{n \in \mathbb{Z}} I_\pm^{(n)}(\tau, x) (-z)^n, \text{ s.t. } (\mathbf{f}_\tau^\pm, \mathbf{1}_\alpha) := \pm \sum_{n \in \mathbb{Z}} D_x^n (k^{-1} \phi_\alpha(x) x^{-k}),$$

where $\alpha \in \mathbb{Z}_k \sqcup \mathbb{Z}_m$. Let us compute $I_\pm^{(0)}(\tau, x)$. Note that the terms in the above sum which contribute to $I_\pm^{(0)}$ are the ones with $n \geq 0$. The rest, according to Lemma 4.2, do not contribute. Thus

$$(4.9) \quad (I_\pm^{(0)}(\tau, x), \mathbf{1}_\alpha) = \phi_\alpha(x) \frac{\omega}{df_\tau}.$$

Note that $D_x \mathbf{f}_\tau^\pm = \mathbf{f}_\tau^\pm$, thus by comparing the coefficients in front of $(-z)^{n+1}$ we get the following recursive relation:

$$(4.10) \quad \partial_x I^{(n)}(\tau, x) = (\partial_x f_\tau) I^{(n+1)}(\tau, x).$$

In particular, all coefficients $I^{(n)}$ are rational vector-valued functions on Y_τ with possible poles only at the critical points of f_τ .

In a neighborhood of $x = \infty$ we choose another (formal) coordinate $\lambda = x + a_0 + a_1 x^{-1} + \dots$ such that λ is a formal solution to the equation

$$(4.11) \quad \lambda^k + \nu_1 \log \lambda + \nu_0 \log(Q/\lambda) = f_\tau(x),$$

where $f_\tau(x) = f(x, Qe^\tau/x)$.

We will show that $S_\tau \mathbf{f}^\pm(\lambda) = \mathbf{f}_\tau^\pm(x)$. It is enough to prove that

$$\sum_{n \in \mathbb{Z}} D_x^n (k^{-1} \phi_\alpha(x) x^{-k}) = (\mathbf{f}_\tau^\pm(x), \mathbf{1}_\alpha) = (S_\tau \mathbf{f}^\pm(\lambda), \mathbf{1}_\alpha) = (\partial_\alpha J, \mathbf{f}^\pm(\lambda)),$$

where $\alpha \in \mathbb{Z}_k \sqcup \mathbb{Z}_m$, ∂_α is the derivative along vector $\mathbf{1}_\alpha$, and the last equality is deduced after comparing the definitions of the J -function J and S_τ of $\mathbb{CP}_{k,m}^1$.

Let us compute $(\partial_\alpha J, \mathbf{f}^\pm)$. Assume first that $\alpha = i'/k, 1 \leq i' \leq k$. Note that only the first sum in the formula for $z\partial_\alpha J$ (see Corollary 3.4) will contribute to the inner product. Take the d -th summand in this sum. It will have a non-zero pairing only with those terms in $\mathbf{f}^\pm(\lambda)$, (see (1.1)) which correspond to $n \in \mathbb{Z}$ and $i, 1 \leq i \leq k$ s.t.

$$-dm + i' + k - i = 0 \pmod{k}, \quad \text{i.e.,} \quad i = -dm + i' \pmod{k}.$$

Pick $n \in \mathbb{Z}$ in such a way that the product in \mathbf{f}^\pm corresponding to n and i cancels with the product in the d -th summand, i.e., $-i/k + n = (dm - i')/k$. On the other hand, note that the sum of all terms in \mathbf{f}^\pm which have a non-zero pairing with the d -th summand can be written as follows:

$$\sum_{n' \in \mathbb{Z}} \tilde{D}_\lambda^{n'} \frac{\prod_{l=-\infty}^n (\nu + (-i/k + l)z)}{\prod_{l=-\infty}^0 (\nu + (-i/k + l)z)} \lambda^{-(n+1)k+i} \mathbf{1}_{(k-i)/k},$$

where

$$\tilde{D}_\lambda = -z \frac{1}{k} \lambda^{1-k} \partial_\lambda + \frac{1}{k} (\nu_0 - \nu_1) \lambda^{-k}.$$

Thus the pairing between $z\partial_\alpha J$ and \mathbf{f}^\pm is

$$g_\alpha e^{\tau\nu_0/z} \sum_{n' \in \mathbb{Z}} D_\lambda^{n'} \sum_{d \geq 0} \lambda^{-k+i'} \frac{1}{d!} [z^{-1} (Qe^\tau/\lambda)^m]^d = e^{[(Qe^\tau/\lambda)^m + \tau\nu_0]z^{-1}} \sum_{n' \in \mathbb{Z}} D_\lambda^{n'} g_\alpha \lambda^{-k+i'},$$

where D_λ is given by formula (4.8). We recall Proposition 4.1, the change (4.11) and since $g_\alpha \lambda^{i'} = k^{-1} \phi_\alpha(\lambda)$ we get exactly what we wanted to prove. The case when $\alpha = j/m, 1 \leq j \leq m$ is similar and will be omitted.

A similar statement holds for the other vertex operators $\bar{\Gamma}^\pm$. Let $y = Qe^{t_N}/x$ be another coordinate on the complex circle. Put $f_\tau(y) = f(Qe^\tau/y, y)$ and let D_y be the differential operator (4.4) corresponding to $\phi = \partial_y f_\tau$, i.e.,

$$(4.12) \quad D_y = -z \frac{1}{m} y^{1-m} \partial_y + \frac{1}{m} (\nu_1 - \nu_0) y^{-m} + \frac{k}{m} (Qe^\tau)^k y^{-k-m}.$$

We define a vector in the symplectic loop space \mathcal{H}

$$\bar{\mathbf{f}}_\tau^\pm = \sum_{n \in \mathbb{Z}} \bar{I}_\pm^{(n)}(\tau, y) (-z)^n, \text{ s.t. } (\bar{\mathbf{f}}_\tau^\pm, \mathbf{1}_\alpha) := \pm \sum_{n \in \mathbb{Z}} D_y^n (m^{-1} \phi_\alpha(y) y^{-m}).$$

Just like before we prove that the 0-mode is given by

$$(4.13) \quad (\bar{I}_\pm^{(0)}(\tau, y), \mathbf{1}_\alpha) = \phi_\alpha(y) \frac{\omega}{df_\tau}.$$

and that the following recursive relation holds:

$$(4.14) \quad \partial_y \bar{I}^{(n)}(\tau, y) = (\partial_y f_\tau) \bar{I}^{(n+1)}(\tau, y).$$

In a neighborhood of $y = \infty$ we choose a formal coordinate λ such that

$$(4.15) \quad \lambda^m + \nu_0 \log \lambda + \nu_1 \log(Q/\lambda) = f_\tau(y).$$

Then $S_\tau \bar{\mathbf{f}}^\pm(\lambda) = \bar{\mathbf{f}}_\tau^\pm(y)$.

5. FROM DESCENDANTS TO ANCESTORS

Let us describe the HQE which one obtains after conjugating the HQE in Theorem 1.1 by S_τ and then we will give the details of the computation.

An *asymptotical function* is, by definition, an expression

$$\mathcal{T} = \exp \left(\sum_{g=0}^{\infty} \epsilon^{2g-2} \mathcal{T}^{(g)}(\mathbf{t}; Q) \right),$$

where $\mathcal{T}^{(g)}$ are formal series in the sequence of vector variables t_0, t_1, t_2, \dots with coefficients in the Novikov ring $\mathbb{C}[[Q]]$. Furthermore, \mathcal{T} is called *tame* if

$$\left. \frac{\partial}{\partial t_{k_1}^{\alpha_1} \dots \partial t_{k_r}^{\alpha_r}} \right|_{\mathbf{t}=0} \mathcal{T}^{(g)} = 0 \quad \text{whenever} \quad k_1 + k_2 + \dots + k_r > 3g - 3 + r,$$

where t_k^α are the coordinates of t_k with respect to $\{\mathbf{1}_\alpha\}$. We will say that a tame asymptotical function \mathcal{T} satisfies the HQE below if for each integer r

$$(5.1) \quad (\text{res}_{x=0} + \text{res}_{x=\infty}) \, c_r(\tau, x) \, (\Gamma_\tau^- \otimes \Gamma_\tau^+) (\mathcal{T} \otimes \mathcal{T}) \, dx = 0,$$

where Γ_τ^\pm are the vertex operators $e^{\hat{\mathbf{f}}_\tau^\pm}$ (see subsection 4.3) and

$$c_r(\tau, x) = x^{-r-1} \exp \left((r-1) \frac{x^k}{\nu_0 - \nu_1} + (r+1) \frac{(Qe^\tau/x)^m}{\nu_0 - \nu_1} \right).$$

The Hirota quadratic equations (5.1) are interpreted as follows: switch to new variables \mathbf{x} and \mathbf{y} via the substitutions: $\mathbf{q}' = \mathbf{x} + \epsilon \mathbf{y}$, $\mathbf{q}'' = \mathbf{x} - \epsilon \mathbf{y}$. Due to the tameness ([14], section 8, Proposition 6), after canceling the terms independent of x , the 1-form on the LHS of (5.1) expands into a power series in \mathbf{y} and ϵ , such that each coefficient depends polynomially on finitely many $I^{(n)}(\tau, x)$ and finitely many partial derivatives of \mathcal{T} . The residues in (5.1) are interpreted as the residues of meromorphic 1-forms.

According to A. Givental ([16], section 8), the asymptotical function $\mathcal{A}_\tau^{\text{Fr}} := (\Psi \text{Re}^{U/z}) \hat{\prod} \mathcal{D}_{\text{pt}}(\mathbf{q}^i)$ is tame. Slightly abusing the notations we use $\tau \in \mathbb{C}$ to denote also the cohomology class τp . The goal in this section is to prove the following theorem.

Theorem 5.1. \mathcal{D}^{Fr} satisfies (1.2) iff $\mathcal{A}_\tau^{\text{Fr}}$ satisfies (5.1).

Proof. We recall formula (4.1) and the main result in subsection 4.3:

$$\hat{S}_\tau \Gamma^\pm \hat{S}_\tau^{-1} = e^{W/2} \Gamma_\tau^\pm, \quad \hat{S}_\tau \bar{\Gamma}^\pm \hat{S}_\tau^{-1} = e^{\bar{W}/2} \bar{\Gamma}_\tau^\pm,$$

where

$$W = W_\tau((\mathbf{f}^+(\lambda))_+, (\mathbf{f}^+(\lambda))_+), \quad \overline{W} = W_\tau((\overline{\mathbf{f}}^+(\lambda))_+, (\overline{\mathbf{f}}^+(\lambda))_+).$$

We will prove that

$$(5.2) \quad W = C + \frac{2}{\nu_0 - \nu_1} (Qe^\tau/x)^m + \log \frac{\lambda(k\lambda^k + \nu_1 - \nu_0)}{x^2 \partial_x f_\tau},$$

$$(5.3) \quad \overline{W} = \overline{C} + \frac{2}{\nu_1 - \nu_0} (Qe^\tau/y)^k + \log \frac{\lambda(m\lambda^m + \nu_0 - \nu_1)}{y^2 \partial_y f_\tau}$$

where $C = (S_1 \mathbf{1}_{0/k}, \mathbf{1}_{0/k})$, $\overline{C} = (S_1 \mathbf{1}_{0/m}, \mathbf{1}_{0/m})$, and x and y are related to λ respectively via (4.11) and (4.15). It is sufficient to establish the first formula, because for the second one one just interchange k with m , ν_0 with ν_1 , and x with y . Using that $\partial_x I_+^{(k)} = (\partial_x f_\tau) I_+^{(k+1)}$ we get

$$\begin{aligned} \partial_x W &= \partial_x W_\tau((\mathbf{f}^+)_+, (\mathbf{f}^+)_+) = \partial_x \sum_{n,l \geq 0} (W_{nl} I_+^{(l)}, I_+^{(n)}) (-1)^{n+l} = \\ &= - \sum_{n,l \geq 0} \left[([W_{n,l-1} + W_{n-1,l}] I_+^{(l)}, I_+^{(n)}) \right] (\partial_x f_\tau) (-1)^{n+l} = \\ &= - \sum_{n,l \geq 0} \left[(S_l (-1)^l I_+^{(l)}, S_n (-1)^n I_+^{(n)}) - (I_+^{(0)}(\lambda), I_+^{(0)}(\lambda)) \right] (\partial_x f_\tau) = \\ &= \left[- \left(I_+^{(0)}(\tau, x), I_+^{(0)}(\tau, x) \right) + \left(I_+^{(0)}(\lambda), I_+^{(0)}(\lambda) \right) \right] (\partial_x f_\tau), \end{aligned}$$

where λ and x are related via equation (4.11). The two 1-forms $\left(I_+^{(0)}(\tau, x), I_+^{(0)}(\tau, x) \right) df_\tau$ and $\left(I_+^{(0)}(\lambda), I_+^{(0)}(\lambda) \right) df_\tau$ are equal respectively to

$$(k\nu^{-1}x^{2k} + m\overline{\nu}^{-1}(Qe^\tau/x)^{2m} + k(k-1)x^k + m(m-1)(Qe^\tau/x)^m) \frac{dx}{x^2 f'_\tau}$$

and

$$((k-1)\lambda^k + \nu^{-1}\lambda^{2k}) \frac{d\lambda}{\lambda(\lambda^k - \nu)}.$$

One can check that primitives of these two 1-forms are given respectively by

$$(5.4) \quad \log(x^2 f'_\tau) + \frac{f_\tau(x) - 2(Qe^\tau/x)^m}{\nu_0 - \nu_1} \quad \text{and} \quad \frac{\lambda^k}{\nu_0 - \nu_1} + \log(\lambda^k - \nu).$$

In order to fix the integration constant C , note that $(\mathbf{f}^+)_+ = \mathbf{1}_{0/k}$ for $x = \infty$. Thus

$$C = W|_{x=\infty} = W_\tau(\mathbf{1}_{0/k}, \mathbf{1}_{0/k}) = (W_{0,0} \mathbf{1}_{0/k}, \mathbf{1}_{0/k}) = (S_1 \mathbf{1}_{0/k}, \mathbf{1}_{0/k}).$$

The rest of the proof follows the argument in section 3.5 in [19]. Since $\mathcal{D} = \widehat{S_\tau}^{-1} \mathcal{A}_\tau$ up to a prefactor, we get that \mathcal{D} satisfies (1.2) iff \mathcal{A}_τ satisfies the following HQE:

$$(5.5) \quad \text{res}_{\lambda=\infty} \frac{d\lambda}{\lambda} \left(\lambda^{n-l} e^W \Gamma_\tau^- \otimes \Gamma_\tau^+ - (Q/\lambda)^{n-l} e^{\overline{W}} \overline{\Gamma}_\tau^+ \otimes \overline{\Gamma}_\tau^- \right) \\ \left(e^{(n+1)\hat{\phi}_{0/k}(\tau,z)+n\hat{\phi}_{0/m}(\tau,z)} \otimes e^{l\hat{\phi}_{0/k}(\tau,z)+(l+1)\hat{\phi}_{0/m}(\tau,z)} \right) (\mathcal{A}_\tau \otimes \mathcal{A}_\tau) = 0,$$

where $\phi_{0/k} = S_\tau \mathbf{1}_{0/k}$ and $\phi_{0/m} = S_\tau \mathbf{1}_{0/m}$. This is the place where we will use that \mathcal{A}_τ is a tame asymptotical function. The tameness condition implies that after the substitutions $\epsilon \mathbf{y} = (\mathbf{q}' - \mathbf{q}'')/2$ and $\mathbf{x} = (\mathbf{q}' + \mathbf{q}'')/2$, and the cancellation of terms that do not depend on λ , the 1-form in (5.5) becomes a formal series in \mathbf{y} and ϵ with coefficients depending polynomially on finitely many of the modes $I_\pm^{(n)}$ and $\overline{I}_\pm^{(n)}$ and finitely many partial derivatives of $\overline{\mathcal{F}}_\tau(\mathbf{x}) := \log \mathcal{A}_\tau$. Furthermore, if we choose two new (formal) coordinates x and y in a neighborhood of $\lambda = \infty$ according to (4.11) and (4.15) then the coefficients $I_\pm^{(n)}$ and $\overline{I}_\pm^{(n)}$ become rational functions respectively in x and y . Thus the LHS of (5.5) is a formal series in \mathbf{x}, \mathbf{y} , and ϵ whose coefficients are residues of rational 1-forms. In particular, the action of the translation operator

$$e^{-(n+1)\hat{\phi}_{0/k}(\tau,z)-n\hat{\phi}_{0/m}(\tau,z)} \otimes e^{-l\hat{\phi}_{0/k}(\tau,z)-(l+1)\hat{\phi}_{0/m}(\tau,z)}$$

on (5.5) is well defined, i.e., we can cancel the corresponding term in (5.5). However, since $e^{\hat{f}} e^{\hat{g}} = e^{\Omega(f,g)} e^{\hat{g}} e^{\hat{f}}$, the two vertex-operator terms in (5.5) will gain the following commutation factors:

$$e^{\Omega(-(n+1)\phi_{0/k}-n\phi_{0/m}, \mathbf{f}_\tau^-) + \Omega(-l\phi_{0/k}-(l+1)\phi_{0/m}, \mathbf{f}_\tau^+)} = e^{(l-n-1)\frac{\lambda^k}{\nu_0-\nu_1}}$$

and

$$e^{\Omega(-(n+1)\phi_{0/k}-n\phi_{0/m}, \overline{\mathbf{f}}_\tau^+) + \Omega(-l\phi_{0/k}-(l+1)\phi_{0/m}, \overline{\mathbf{f}}_\tau^-)} = e^{(n-l-1)\frac{\lambda^m}{\nu_1-\nu_0}},$$

where we used that S_τ is a symplectic transformation, thus

$$\Omega(S_\tau \mathbf{1}_\alpha, \mathbf{f}_\tau^\pm) = \Omega(S_\tau \mathbf{1}_\alpha, S_\tau \mathbf{f}^\pm) = \Omega(\mathbf{1}_\alpha, \mathbf{f}^\pm), \quad \alpha \in \mathbb{Z}_k \sqcup \mathbb{Z}_m,$$

and the later is easy to compute from formula (1.1). Thus (5.5) is equivalent to the following HQE:

$$\text{res}_{\lambda=\infty} \frac{d\lambda}{\lambda} \times \\ \left(\lambda^{-r} e^{W+(r-1)\frac{\lambda^k}{\nu_0-\nu_1}} \Gamma_\tau^- \otimes \Gamma_\tau^+ - \left(\frac{Q}{\lambda} \right)^{-r} e^{\overline{W}-(r+1)\frac{\lambda^m}{\nu_1-\nu_0}} \overline{\Gamma}_\tau^+ \otimes \overline{\Gamma}_\tau^- \right) (\mathcal{A}_\tau \otimes \mathcal{A}_\tau) = 0,$$

where we put $r = l - n$. We write the above residue sum as a difference of two residues. In the first one we change from λ to x according to (4.11) and we

recall formula (5.2). After a short computation we get:

$$(5.6) \quad \text{res}_{x=\infty} e^{C+(r-1)\tau \frac{\nu_0}{\nu_0-\nu_1}} c_r(\tau, x) (\Gamma_\tau^- \otimes \Gamma_\tau^+) (\mathcal{A}_\tau \otimes \mathcal{A}_\tau) dx,$$

where $c_r(\tau, x)$ is the same as in (5.1). In the second residue we change λ to y according to (4.15) and we recall formula (5.3):

$$\text{res}_{y=\infty} e^{\overline{C}+(r+1)\tau \frac{\nu_0}{\nu_0-\nu_1}} c_r(\tau, Qe^\tau/y) (\overline{\Gamma}_\tau^+ \otimes \overline{\Gamma}_\tau^-) (\mathcal{A}_\tau \otimes \mathcal{A}_\tau) Q \frac{dy}{y^2}.$$

Note that if we change $y = Qe^\tau/x$ then $\overline{\Gamma}_\tau^\pm = \Gamma_\tau^\mp$, thus the last residue transforms into

$$(5.7) \quad - \text{res}_{x=0} e^{\overline{C}+(r+1)\tau \frac{\nu_0}{\nu_0-\nu_1} - \tau} c_r(\tau, x) (\Gamma_\tau^- \otimes \Gamma_\tau^+) (\mathcal{A}_\tau \otimes \mathcal{A}_\tau) dx.$$

We compare (5.6) and (5.7) and we see that in order to finish the proof of the theorem, we just need to verify that

$$C + (r-1) \frac{\tau \nu_0}{\nu_0 - \nu_1} = \overline{C} + (r+1) \frac{\tau \nu_0}{\nu_0 - \nu_1} - \tau, \text{ i.e. } C - \overline{C} = \tau \frac{\nu_0 + \nu_1}{\nu_0 - \nu_1}.$$

On the other hand we know that $C = (S_1 \mathbf{1}_{0/k}, \mathbf{1}_{0/k})$ which is equal to the coefficient in front of z^{-1} in $(\partial_{0/k} J, \mathbf{1}_{0/k})$. The later can be computed from Corollary 3.4. The answer is the following:

$$C = \begin{cases} \tau \nu_0 / (\nu_0 - \nu_1) & \text{if } k \neq m, \\ \tau \nu_0 / (\nu_0 - \nu_1) + k(Qe^\tau)^k / (\nu_0 - \nu_1)^2 & \text{if } k = m. \end{cases}$$

Similarly,

$$\overline{C} = \begin{cases} \tau \nu_1 / (\nu_1 - \nu_0) & \text{if } k \neq m, \\ \tau \nu_1 / (\nu_1 - \nu_0) + m(Qe^\tau)^m / (\nu_1 - \nu_0)^2 & \text{if } k = m. \end{cases}$$

The theorem follows. \square

6. FROM ANCESTORS TO KDV

In this section we prove that the ancestor potential $\mathcal{A}_\tau^{\text{Fr}}$ satisfies (5.1). In view of Theorem 5.1 this would imply Theorem 1.1. Note that the vertex operators Γ_τ^\pm have poles only at $x = 0, \infty$, or ξ_i , $1 \leq i \leq k + m$, where the later are the critical points of f_τ . Thus it is enough to prove that the residue of the 1-form in (5.1) at each critical point ξ_i is 0.

Let us fix a critical point ξ_i and denote by $u_i = f_\tau(\xi_i)$ the corresponding critical value. The function f_τ induces a map between a neighborhood of $x = \xi_i$ and a neighborhood of $\Lambda = u_i$ which is a double covering branched at u_i . We pick a reference point Λ_0 in a neighborhood of u_i and denote by $x_\pm(\Lambda_0)$ the two points which cover Λ_0 . Finally, let us denote by $x_\pm(\Lambda)$ the points covering Λ . Note that $x_\pm(\Lambda)$ depend on a choice of a path C between Λ_0 and Λ avoiding

u_i . On the other hand, for any function $g(x)$ meromorphic in a neighborhood of ξ_i we have

$$\operatorname{res}_{x=\xi_i} g(x)dx = \operatorname{res}_{\Lambda=u^i} \sum_{\pm} g(x_{\pm}(\Lambda)) \frac{\partial x_{\pm}}{\partial \Lambda}(\Lambda) d\Lambda.$$

Thus the vanishing of the residue at ξ_i of (5.1) is equivalent to:

$$(6.1) \quad \operatorname{res}_{\Lambda=u^i} \left\{ d\Lambda \sum_{\pm} \frac{e^{\frac{2}{\nu_0-\nu_1}(Qe^{\tau}/x_{\pm})^m}}{x_{\pm}^2 f'_{\tau}(x_{\pm})} (\Gamma_{\tau}^{\beta_{\pm}} \otimes \Gamma_{\tau}^{-\beta_{\pm}})(\mathcal{A}_{\tau} \otimes \mathcal{A}_{\tau}) \right\} e^{\frac{r-1}{\nu_0-\nu_1}\Lambda} = 0,$$

where β_{\pm} are the one point cycles $[x_{\pm}(\Lambda)] \in H^0(f_{\tau}^{-1}(\Lambda); \mathbb{Z})$ and the vertex operators can be described as follows:

$$\mathbf{f}_{\tau}^{\pm\beta}(\Lambda) = - \int_{\beta} \mathbf{f}_{\tau}^{\pm}(x), \quad \Gamma_{\tau}^{\pm\beta} = \left(e^{\mathbf{f}_{\tau}^{\pm\beta}} \right)^{\wedge}, \quad \beta \in H^0(f_{\tau}^{-1}(\Lambda); \mathbb{Z}).$$

We will prove that the 1-form in the $\{ \}$ -brackets in (6.1) is analytic in Λ . In particular this would imply that the residue (6.1) is 0. The proof follows closely the argument in [16].

Note that the vector-valued function $I_{\beta_+-\beta_-}^{(0)}(\tau, \Lambda)$ can be expanded in a neighborhood of $\Lambda = u_i$ as follows

$$(6.2) \quad I_{(\beta_+-\beta_-)/2}^{(0)}(\tau, \Lambda) = \frac{1}{\sqrt{2(\Lambda - u^i)}} (e_i + O(\Lambda - u^i)),$$

where the standard vector e_i in \mathbb{C}^{k+m} is identified via Ψ with a vector in H and the value of $\sqrt{2(\Lambda - u_i)}$ is fixed as follows. Choose a path C_0 from $u_i + 1$ to Λ_0 , then the translation of $C \circ C_0$ along vector $-u_i$ is a path from 1 to $\Lambda - u_i$. If we choose C_0 arbitrary then (6.2) is correct up to a sign, so if necessary change C_0 in order to achieve equality. We introduce also a 1-form $\mathcal{W}_{\beta', \beta''}$, called *the phase form*, defined as follows:

$$\mathcal{W}_{\beta', \beta''} = - \left(I_{\beta'}^{(0)}(\tau, \Lambda), I_{\beta''}^{(0)}(\tau, \Lambda) \right) d\Lambda, \quad \beta', \beta'' \in H_0(f_{\tau}^{-1}(\Lambda); \mathbb{Q}).$$

Lemma A. *The vertex operators $\Gamma_{\tau}^{\beta_{\pm}}$ and $\Gamma_{\tau}^{-\beta_{\pm}}$ factor as follows:*

$$\Gamma_{\tau}^{\beta_{\pm}} = e^{\pm K} \Gamma_{\tau}^{(\beta_{\pm}+\beta_{\mp})/2} \Gamma_{\tau}^{\pm(\beta_{\pm}-\beta_{\mp})/2}, \quad \Gamma_{\tau}^{-\beta_{\pm}} = e^{\pm K} \Gamma_{\tau}^{-(\beta_{\pm}+\beta_{\mp})/2} \Gamma_{\tau}^{\mp(\beta_{\pm}-\beta_{\mp})/2},$$

where

$$K = \int_{\Lambda}^{u_i} \mathcal{W}_{(\beta_+-\beta_-)/2, (\beta_++\beta_-)/2}.$$

Proof. This is Proposition 4 from [16], section 7. □

Lemma B. *For Λ near the critical value u_i , the following formula holds:*

$$(6.3) \quad \Gamma_{\tau}^{\pm(\beta_+-\beta_-)/2} (\Psi R e^{U/z})^{\wedge} = e^{(W_i+u_i)/2} (\Psi R e^{U/z})^{\wedge} \Gamma_{\tau}^{\pm},$$

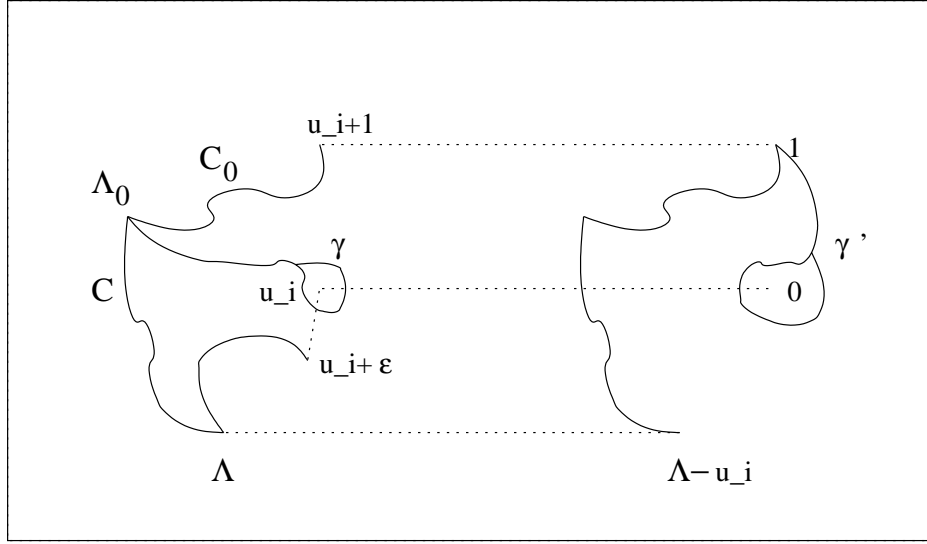


FIGURE 1. Integration paths.

where

$$W_i = \lim_{\epsilon \rightarrow 0} \int_{\Lambda}^{u_i + \epsilon} \left(\mathcal{W}_{(\beta_+ - \beta_-)/2, (\beta_+ - \beta_-)/2} + \frac{d\xi}{2(\xi - u_i)} \right), \quad w_i = - \int_{\Lambda - u_i}^{\Lambda} \frac{d\xi}{2\xi},$$

$$\Gamma^{\pm} = \exp \left(\sum_{n \in \mathbb{Z}} (-z \partial_{\Lambda})^n \frac{e_i}{\pm \sqrt{2\Lambda}} \right).$$

Proof. This is Theorem 3 from [16]. \square

The integration path in the definition of W_i is any path connecting Λ and $u_i + \epsilon$ and $\epsilon \rightarrow 0$ in such a way that $u_i + \epsilon \rightarrow u_i$ along a straight segment. The integration path in w_i is the straight segment connecting $\Lambda - u_i$ and Λ . The various integration paths are depicted on Figure 1.

Using Lemma A and Lemma B we get that the expression in the $\{ \}$ -brackets in (6.1) is equal to

$$\Gamma_{\tau}^{(\beta_+ + \beta_-)/2} \otimes \Gamma_{\tau}^{-(\beta_+ + \beta_-)/2} (\Psi Re^{U/z})^{\wedge} \otimes (\Psi Re^{U/z})^{\wedge}$$

$$\left\{ \sum_{\pm} c_{\pm}(\tau, \Lambda) \Gamma_{(i)}^{\pm} \otimes \Gamma_{(i)}^{\mp} \frac{d\Lambda}{\pm \sqrt{\Lambda}} \right\} \prod_{j=1}^{k+m} \mathcal{D}_{\text{pt}}(\mathbf{q}^j) \otimes \prod_{j=1}^{k+m} \mathcal{D}_{\text{pt}}(\mathbf{q}^j),$$

where the index i in $\Gamma_{(i)}^{\pm}$ is just to emphasize that the vertex operator is acting on the i -th factor in the product $\prod_{j=1}^{k+m} \mathcal{D}_{\text{pt}}(\mathbf{q}^j)$ and the coefficients c_{\pm} are given

by the following formula:

$$(6.4) \quad \log c_{\pm} = 2 \frac{(Qe^t/x_{\pm})^m}{\nu_0 - \nu_1} - \log(x_{\pm}^2 f'_{\tau}(x_{\pm})) + W_i + w_i \pm 2K + \int_{\gamma_{\pm}} \frac{d\xi}{2\xi},$$

where the path γ_+ is the composition of $C \circ C_0$ and the line segment from 1 to $u_i + 1$ and $\gamma_- = \gamma_+ \circ \gamma'$, where γ' is a simple loop around 0 starting and ending at 1 (see Figure 1).

We will prove that with respect to Λ the functions c_+ and c_- are analytic and coincide in a neighborhood of u_i . This would finish the proof of the theorem because, according to A. Givental [16], the 1-form

$$\sum_{\pm} \Gamma_{(i)}^{\pm} \otimes \Gamma_{(i)}^{\mp} \frac{d\Lambda}{\pm\sqrt{\Lambda}} \mathcal{T} \otimes \mathcal{T}$$

is analytic in Λ whenever \mathcal{T} is a tau-function of the KdV hierarchy. On the other hand, according to M. Kontsevich [18], \mathcal{D}_{pt} is a tau-function of the KdV hierarchy, thus the theorem would follow.

Note that the first two terms in (6.4), up to a summand of $\Lambda/(\nu_0 - \nu_1)$, coincide with the primitive (see (5.4)) of the 1-form $\mathcal{W}_{\beta_{\pm}, \beta_{\pm}}$. Thus

$$2 \frac{(Qe^t/x_{\pm}(\Lambda))^m}{\nu_0 - \nu_1} - \log(x_{\pm}^2 f'_{\tau}) = \int_{\Lambda_0}^{\Lambda} \mathcal{W}_{\beta_{\pm}, \beta_{\pm}} + \Lambda/(\nu_0 - \nu_1) + C_{\pm}$$

where the constants C_{\pm} are independent of Λ (they depend only on $x_{\pm}(\Lambda_0)$). and their difference can be interpreted as

$$C_+ - C_- = \oint_{\gamma} \mathcal{W}_{\beta_-, \beta_-},$$

where γ is a simple loop around u_i (see Figure 1). Therefore $\log c_{\pm}$ admits the following integral presentation

$$\begin{aligned} \log c_{\pm} &= \lim_{\epsilon \rightarrow 0} \\ &\left(\int_{\Lambda_0}^{\Lambda} \mathcal{W}_{\beta_{\pm}, \beta_{\pm}} + \int_{\Lambda}^{u_i + \epsilon} \mathcal{W}_{(\beta_+ - \beta_-)/2, (\beta_+ - \beta_-)/2} \pm 2 \int_{\Lambda}^{u_i + \epsilon} \mathcal{W}_{(\beta_+ - \beta_-)/2, (\beta_+ + \beta_-)/2} + \right. \\ &\left. + \int_{\Lambda}^{u_i + \epsilon} \frac{d\xi}{2(\xi - u_i)} - \int_{\Lambda - u_i}^{\Lambda} \frac{d\xi}{2\xi} + \int_{\gamma_{\pm}} \frac{d\xi}{2\xi} + \frac{1}{\nu_0 - \nu_1} \Lambda + C_{\pm} \right). \end{aligned}$$

In the first integral put $\beta_{\pm} = (\beta_{\pm} + \beta_{\mp})/2 + (\beta_{\pm} - \beta_{\mp})/2$. After a simple computation we get:

$$\begin{aligned} \log c_{\pm} &= \int_{\Lambda_0}^{\Lambda} \mathcal{W}_{(\beta_{\pm} + \beta_{\mp})/2, (\beta_{\pm} + \beta_{\mp})/2} + \frac{1}{\nu_0 - \nu_1} \Lambda + C_{\pm} \\ \lim_{\epsilon \rightarrow 0} &\left(\int_{\Lambda_0}^{u_i + \epsilon} \mathcal{W}_{(\beta_+ - \beta_-)/2, (\beta_+ - \beta_-)/2} + 2 \int_{\Lambda_0}^{u_i + \epsilon} \mathcal{W}_{(\beta_{\pm} - \beta_{\mp})/2, (\beta_{\pm} + \beta_{\mp})/2} + \int_{\gamma'_{\pm}} \frac{d\xi}{2\xi} \right), \end{aligned}$$

where γ'_\pm is the composition of the paths: γ_\pm – starting at 1 and ending at Λ , the straight segment between Λ and $\Lambda - u_i$ (i.e., the integration path for w_i), and the path from $\Lambda - u_i$ to ϵ obtained by translating the path between Λ and $u_i + \epsilon$. Furthermore, we rewrite the last formula as follows:

$$\int_{u_i}^{\Lambda} \mathcal{W}_{(\beta_\pm + \beta_\mp)/2, (\beta_\pm + \beta_\mp)/2} + \frac{1}{\nu_0 - \nu_1} \Lambda + C_\pm + \lim_{\epsilon \rightarrow 0} \left(\int_{\Lambda_0}^{u_i + \epsilon} \mathcal{W}_{\beta_\pm, \beta_\pm} + \int_{\gamma'_\pm} \frac{d\xi}{2\xi} \right).$$

The first integral is analytic near $\Lambda = u_i$, because near $\Lambda = u_i$, the mode $I_{\beta_\pm}^{(0)}$ expands as a Laurent series in $\sqrt{\Lambda - u_i}$ with singular term at most $1/\sqrt{(\Lambda - u_i)}$. However the analytical continuation around $\Lambda = u_i$ transforms $I_{\beta_\pm}^{(0)}$ into $I_{\beta_\mp}^{(0)}$, hence $I_{\beta_\pm}^{(0)} + I_{\beta_\mp}^{(0)}$ must be single-valued and in particular, it could not have singular terms. Since the limit is independent of Λ , the analyticity of c_\pm follows.

It remains to prove that c_+ and c_- are equal.

$$\log c_+ - \log c_- = \lim_{\epsilon \rightarrow 0} \left\{ \oint_{\gamma_\epsilon} \mathcal{W}_{\beta_-, \beta_-} + \oint_{\gamma'} \frac{d\xi}{2\xi} \right\},$$

where γ_ϵ is a closed loop around u_i starting and ending at $u_i + \epsilon$. The second integral is $\pm\pi i$ (the sign depends on the orientation of the loop γ'). To compute the first one, write $\beta_- = (\beta_- - \beta_+)/2 + (\beta_- + \beta_+)/2$ and transform the integrand into

$$\left(I_{(\beta_- - \beta_+)/2}^{(0)}, I_{(\beta_- - \beta_+)/2}^{(0)} \right) + 2 \left(I_{(\beta_- - \beta_+)/2}^{(0)}, I_{(\beta_- + \beta_+)/2}^{(0)} \right) + \left(I_{(\beta_- + \beta_+)/2}^{(0)}, I_{(\beta_- + \beta_+)/2}^{(0)} \right).$$

The last term does not contribute to the integral because it is analytic in Λ . The middle one, up to a factor analytic in Λ , coincides with $(\Lambda - u_i)^{-1/2}$, therefore its integral along γ_ϵ vanishes in the limit $\epsilon \rightarrow 0$. Finally, the first term has an expansion of the type

$$\left(I_{(\beta_- - \beta_+)/2}^{(0)}, I_{(\beta_- - \beta_+)/2}^{(0)} \right) = \frac{1}{2(\Lambda - u_i)} + O(\Lambda - u_i)$$

and so it contributes only $\pm\pi i$ to the integral. Thus $(\log c_+ - \log c_-)$ is an integer multiple of $2\pi i$, which implies that $c_+ = c_-$.

APPENDIX A. PROOF OF COROLLARY 3.4

A.1. Combinatorial notations. We assume that k, m are co-prime. Without loss of generality we may assume that $k > m$. For each integer i with $1 \leq i \leq m - 1$ we define two positive integers q_i, r_i as follows:

$$ik = mq_i + r_i, \quad \text{where } 0 \leq r_i \leq m - 1.$$

Note that $r_i \neq 0$, otherwise k, m are not co-prime. Also put $q_0 = 0, q_m = k$. Clearly we have

$$\frac{q_i}{k} < \frac{i}{m} < \frac{q_i + 1}{k}.$$

Lemma A.1. $\frac{i+1}{m} > \frac{q_i+1}{k}$.

Proof. The inequality is equivalent to $r_i + k > m$, which follows from the assumption $k > m$. \square

We introduce the sequence $\{s_\alpha\}_{\alpha=1}^{k+m}$ which is a rearrangement of the set¹ of numbers $\{\frac{0}{k}, \frac{1}{k}, \dots, \frac{k-1}{k}, \frac{0}{m}, \frac{1}{m}, \dots, \frac{m-1}{m}\}$ into increasing order: $s_1 = \frac{0}{k}$, and

$$s_\alpha = \begin{cases} \frac{j}{m}, & \text{if } \alpha = j + 2 + \sum_{i=0}^j q_i \\ \frac{q_j+l}{k}, & \text{if } \alpha = j + 2 + l + \sum_{i=0}^j q_i \end{cases}$$

where $0 \leq j \leq m-1$ and $1 \leq l \leq q_{j+1}$.

We define differential operators δ_α by the following rule:

- if $s_\alpha \in \{\frac{0}{k}, \frac{1}{k}, \dots, \frac{k-1}{k}\}$, define

$$\delta_\alpha = \frac{z}{m} \frac{\partial}{\partial \tau} - \frac{\nu_0}{m} - s_\alpha k z.$$

- if $s_\alpha \in \{\frac{0}{m}, \frac{1}{m}, \dots, \frac{m-1}{m}\}$, define

$$\delta_\alpha = \frac{z}{k} \frac{\partial}{\partial \tau} - \frac{\nu_1}{k} - s_\alpha m z.$$

Now put $D_0 = D_1 = id$ and for $\alpha \geq 2$, define

$$D_\alpha := Q^{-mks_\alpha} e^{-mks_\alpha \tau} \prod_{\gamma < \alpha} \delta_\gamma.$$

For $s_\alpha \in \{\frac{0}{k}, \frac{1}{k}, \dots, \frac{k-1}{k}\}$, define $\tilde{s}_\alpha := -ms_\alpha$. In this case we may write $s_\alpha = \frac{q_s-a}{k}$ for some $1 \leq s \leq m$ and $0 \leq a \leq (q_s - q_{s-1}) - 1$. We have

$$0 \leq r_s + am \leq r_s + mq_s - mq_{s-1} - m = sk - ((s-1)k - r_{s-1}) - m = k + r_{s-1} - m < k.$$

Thus the fractional part of $-ms_\alpha = \frac{-sk+r_s+am}{k}$ is $\frac{r_s+am}{k}$.

For $s_\alpha = \frac{s}{m}$, we define $\tilde{s}_\alpha := -ks_\alpha$. We have $-ks = -mq_s - r_s = -m(1 + q_s) + m - r_s$. Thus the fractional part of $-ks_\alpha$ is $\frac{m-r_s}{m}$.

¹Note that we treat $\frac{0}{k}$ and $\frac{0}{m}$ as two *different* numbers.

A.2. Proof of Corollary 3.4. It is straightforward to see that the vector $D_\alpha J_X$ is a linear combination of z times partial derivatives of J_X (restricted to $H^2(\mathbb{CP}_{k,m}^1)$). We will prove the following equalities:

$$(A.1) \quad \delta_1 J_X = (mg_{m/m})^{-1} z \partial_{m/m} J_X.$$

$$(A.2) \quad \delta_2 J_X = (kg_{k/k})^{-1} z \partial_{k/k} J_X.$$

$$(A.3) \quad D_\alpha J_X = z \partial_{\bar{s}_\alpha} J_X, \quad \alpha \geq 3.$$

The proof of (A.1)–(A.3) requires explicit computations of the left-hand sides of them. (3.4)–(3.5) will follow as a by-product.

First we show (A.1). Applying δ_1 to (3.1) yields

$$(A.4) \quad \begin{aligned} \delta_1 J_X = & z e^{\tau \nu_0/z} \sum_{d>0} \frac{Q^{dm} e^{dm\tau} \left(\frac{z}{m} \left(\frac{\nu_0}{z} + dm \right) - \frac{\nu_0}{m} \right)}{d! z^d \prod_{b=\{dm/k\}}^{dm/k} (\nu + bz)} \mathbf{1}_{-dm/k} \\ & + z e^{\tau \nu_1/z} \left(\frac{z}{m} \frac{\nu_1}{z} - \frac{\nu_0}{m} \right) \mathbf{1}_{0/m} + z e^{\tau \nu_1/z} \sum_{d>0} \frac{Q^{dk} e^{dk\tau} \left(\left(\frac{\nu_1}{z} + dk \right) \frac{z}{m} - \frac{\nu_0}{m} \right)}{\prod_{b=\{dk/m\}}^{dk/m} (\bar{\nu} + bz) d! z^d} \mathbf{1}_{-dk/m}. \end{aligned}$$

Here the term with highest power in z is $z \bar{\nu} \mathbf{1}_{0/m}$, hence (A.1) holds. To see (3.5) in the case $j = m$, we rearrange (A.4) as follows:

$$\begin{aligned} & \sum_{d>0} \frac{Q^{dm} e^{dm\tau} \left(\frac{z}{m} \left(\frac{\nu_0}{z} + dm \right) - \frac{\nu_0}{m} \right)}{d! z^d \prod_{b=\{dm/k\}}^{dm/k} (\nu + bz)} \mathbf{1}_{-dm/k} = \sum_{d>0} \frac{Q^{dm} e^{dm\tau} dz}{d! z^d \prod_{b=\{dm/k\}}^{dm/k} (\nu + bz)} \mathbf{1}_{-dm/k} \\ & = \sum_{d \geq 0} \frac{Q^{dm+m} e^{dm\tau+m\tau}}{d! z^d \prod_{b=\{\frac{dm+m}{k}\}}^{\frac{dm+m}{k}} (\nu + bz)} \mathbf{1}_{-(dm+m)/k}, \quad (\text{re-indexing}), \\ & \left(\frac{z}{m} \frac{\nu_1}{z} - \frac{\nu_0}{m} \right) \mathbf{1}_{0/m} + \sum_{d>0} \frac{Q^{dk} e^{dk\tau} \left(\left(\frac{\nu_1}{z} + dk \right) \frac{z}{m} - \frac{\nu_0}{m} \right)}{\prod_{b=\{dk/m\}}^{dk/m} (\bar{\nu} + bz) d! z^d} \mathbf{1}_{-dk/m} \\ & = \bar{\nu} \mathbf{1}_{0/m} + \sum_{d>0} \frac{Q^{dk} e^{dk\tau}}{d! z^d} \frac{\prod_{b<\{\frac{dk-m}{m}\}} (\bar{\nu} + bz)}{\prod_{b \leq \frac{dk-m}{m}} (\bar{\nu} + bz)} \mathbf{1}_{-dk/m} \\ & = \sum_{d \geq 0} \frac{Q^{dk} e^{dk\tau}}{d! z^d} \frac{\prod_{b<\{\frac{dk-m}{m}\}} (\bar{\nu} + bz)}{\prod_{b \leq \frac{dk-m}{m}} (\bar{\nu} + bz)} \mathbf{1}_{\frac{-dk+m}{m}}. \end{aligned}$$

(Note our convention on the fractional part $\{ \}$.)

The proof of (A.2) and (3.4) for the case $i = k$ is similar.

(A.3), (3.4) for $i \neq k$, and (3.5) for $j \neq m$ will be proven together by induction on $\alpha \geq 3$.

Case $\alpha = 3$: We compute

(A.5)

$$\begin{aligned}
\delta_2 \delta_1 J_X &= z \sum_{d>0} \frac{e^{\tau \nu_0/z} Q^{dm} e^{dm\tau} dz \left(\frac{z}{k} \left(\frac{\nu_0}{z} + dm \right) - \frac{\nu_1}{k} \right)}{d! z^d \prod_{b=\{dm/k\}}^{dm/k} (\nu + bz)} \mathbf{1}_{-dm/k} \\
&\quad + z e^{\tau \nu_1/z} \left(\frac{z}{k} \frac{\nu_1}{z} - \frac{\nu_1}{k} \right) \bar{\nu} \mathbf{1}_{0/m} + z \sum_{d>0} \frac{e^{\tau \nu_1/z} Q^{dk} e^{dk\tau} \left(\frac{z}{k} \left(\frac{\nu_1}{z} + dk \right) - \frac{\nu_1}{k} \right) (\bar{\nu} + \frac{dk}{m} z)}{\prod_{b=\{dk/m\}}^{dk/m} (\bar{\nu} + bz) d! z^d} \mathbf{1}_{-dk/m} \\
&= z e^{\tau \nu_0/z} \sum_{d>0} \frac{Q^{dm} e^{dm\tau} dz \left(\nu + \frac{dm}{k} z \right)}{d! z^d \prod_{b=\{dm/k\}}^{dm/k} (\nu + bz)} \mathbf{1}_{-dm/k} \\
&\quad + z e^{\tau \nu_1/z} \sum_{d>0} \frac{Q^{dk} e^{dk\tau} (\bar{\nu} + \frac{dk}{m} z) dz}{\prod_{b=\{dk/m\}}^{dk/m} (\bar{\nu} + bz) d! z^d} \mathbf{1}_{-dk/m}
\end{aligned}$$

Here it is easy to see that the term having the highest power of z is $Q^m e^{m\tau} \mathbf{1}_{-m/k}$. So $Q^{-m} e^{-m\tau} \delta_2 \delta_1 J_X = z \partial_{\frac{k-m}{k}} J_X$, proving the case $\alpha = 3$ of (A.3) (note that $\tilde{s}_3 = \frac{k-m}{k}$). We can further simplify (A.5) as follows:

$$\begin{aligned}
Q^{-m} e^{-m\tau} \delta_2 \delta_1 J_X &= z e^{\tau \nu_0/z} \sum_{d>0} \frac{Q^{dm-m} e^{dm\tau-m\tau} \left(\nu + \frac{dm}{k} z \right)}{(d-1)! z^{d-1} \prod_{b=\{dm/k\}}^{dm/k} (\nu + bz)} \mathbf{1}_{-dm/k} \\
&\quad + z e^{\tau \nu_1/z} \sum_{d>0} \frac{Q^{dk-m} e^{dk\tau-m\tau} \left(\bar{\nu} + \frac{dk}{m} z \right)}{\prod_{b=\{dk/m\}}^{dk/m} (\bar{\nu} + bz) (d-1)! z^{d-1}} \mathbf{1}_{-dk/m} \\
&= z e^{\tau \nu_0/z} \sum_{d \geq 0} \frac{Q^{dm} e^{dm\tau} \left(\nu + \frac{(d+1)m}{k} z \right)}{d! z^d \prod_{b=\{(d+1)m/k\}}^{(d+1)m/k} (\nu + bz)} \mathbf{1}_{-(d+1)m/k} \\
&\quad + z e^{\tau \nu_1/z} \sum_{d \geq 0} \frac{Q^{dk+(k-m)} e^{dk\tau+(k-m)\tau} \left(\bar{\nu} + \frac{(d+1)k}{m} z \right)}{\prod_{b=\{(d+1)k/m\}}^{(d+1)k/m} (\bar{\nu} + bz) (d-1)! z^{d-1}} \mathbf{1}_{-(d+1)k/m} \\
&= z e^{\tau \nu_0/z} \sum_{d \geq 0} \frac{Q^{dm} e^{dm\tau}}{d! z^d} \frac{\prod_{b < \{\frac{dm-(k-m)}{k}\}} (\nu + bz)}{\prod_{b \leq \frac{dm-(k-m)}{k}} (\nu + bz)} \mathbf{1}_{-\frac{dm+(k-m)}{k}} \\
&\quad + z e^{\tau \nu_1/z} \sum_{d \geq 0} \frac{Q^{dk+(k-m)} e^{dk\tau+(k-m)\tau}}{\prod_{b=\{\frac{dk+(k-m)}{m}\}}^{\frac{dk+(k-m)}{m}} (\bar{\nu} + bz) (d-1)! z^{d-1}} \mathbf{1}_{-\frac{dk+(k-m)}{m}}.
\end{aligned}$$

This is exactly the $i = k - m$ case of (3.4), as desired.

Induction step: Now consider $3 \leq \alpha \leq k + m - 1$, suppose that (A.3) and the corresponding (3.4) or (3.5) hold for α . Note that

$$(A.6) \quad D_{\alpha+1} J_X = Q^{-kms_{\alpha+1}} e^{-kms_{\alpha+1}\tau} \delta_{\alpha} (Q^{kms_{\alpha}} e^{kms_{\alpha}\tau} z \partial_{\tilde{s}_{\alpha}} J_X).$$

There are three cases which we handle separately.

Case 1: $s_{\alpha}, s_{\alpha+1} \in \{\frac{0}{k}, \dots, \frac{k-1}{k}\}$.

We have $s_{\alpha+1} = s_{\alpha} + \frac{1}{k}$. We may write $\tilde{s}_{\alpha} = i/k$ for some $1 \leq i \leq k$. According to our discussion at the end of subsection A.1 we have $i > m$ and $\tilde{s}_{\alpha+1} = \frac{i-m}{k}$. Also, $\delta_{\alpha} = \frac{z}{m} \frac{\partial}{\partial \tau} - \frac{\nu_0}{m} - s_{\alpha} k z$. By induction, (3.4) holds for this i . We will prove (A.3) for $\alpha + 1$ and (3.4) for $i - m$.

Using (3.4) for this i we calculate

$$(A.7) \quad \begin{aligned} & \delta_{\alpha} (Q^{kms_{\alpha}} e^{kms_{\alpha}\tau} z \partial_{\tilde{s}_{\alpha}} J_X) \\ &= z e^{\tau\nu_0/z} \sum_{d \geq 0} \frac{Q^{dm+kms_{\alpha}} e^{dm\tau+kms_{\alpha}\tau}}{d! z^d} dz \frac{\prod_{b < \{\frac{dm-i}{k}\}} (\nu + bz)}{\prod_{b \leq \frac{dm-i}{k}} (\nu + bz)} \mathbf{1}_{\frac{-dm+i}{k}} \\ & \quad + z e^{\tau\nu_1/z} \sum_{d \geq 0} \frac{Q^{dk+i+kms_{\alpha}} e^{(dk+i)\tau+kms_{\alpha}\tau}}{\prod_{b=\{\frac{dk+i}{m}\}}^{\frac{dk+i}{m}} (\bar{\nu} + bz) d! z^d} (\bar{\nu} + \frac{dk+i}{m} z) \mathbf{1}_{\frac{-(dk+i)}{m}}. \end{aligned}$$

Here the term having the highest power in z is

$$Q^{m+kms_{\alpha}} e^{m\tau+kms_{\alpha}\tau} \frac{\prod_{b < \{\frac{m-i}{k}\}} (\nu + bz)}{\prod_{b \leq \frac{m-i}{k}} (\nu + bz)} \mathbf{1}_{\frac{-m+i}{k}} = Q^{m+kms_{\alpha}} e^{m\tau+kms_{\alpha}\tau} \mathbf{1}_{\frac{-m+i}{k}},$$

because $\frac{m-i}{k} > -1$. In view of $s_{\alpha+1} = s_{\alpha} + \frac{1}{k}$ and (A.6) this implies (A.3) for $\alpha + 1$. Moreover, we may further simplify (A.7) to obtain:

$$\begin{aligned} & z e^{\tau\nu_0/z} \sum_{d \geq 0} \frac{Q^{dm+m+kms_{\alpha}} e^{dm\tau+(m+kms_{\alpha})\tau}}{d! z^d} \frac{\prod_{b < \{\frac{dm+m-i}{k}\}} (\nu + bz)}{\prod_{b \leq \frac{dm+m-i}{k}} (\nu + bz)} \mathbf{1}_{\frac{-dm-m+i}{k}} \\ & + z e^{\tau\nu_1/z} \sum_{d \geq 0} \frac{Q^{dk+i+kms_{\alpha}} e^{(dk+i+kms_{\alpha})\tau}}{d! z^d} \frac{1}{\prod_{b=\{\frac{dk+(i-m)}{m}\}}^{\frac{dk+(i-m)}{m}} (\bar{\nu} + bz)} \mathbf{1}_{\frac{-(dk+(i-m))}{m}}. \end{aligned}$$

Using $s_{\alpha+1} = s_{\alpha} + \frac{1}{k}$ and removing the factor $Q^{kms_{\alpha+1}} e^{kms_{\alpha+1}\tau}$, we obtain (3.4) for $i - m$, as desired.

Case 2: $s_{\alpha} = \frac{q_s}{k}, s_{\alpha+1} = \frac{s}{m}$.

In this case $\tilde{s}_{\alpha} = \frac{r_s}{k}, \tilde{s}_{\alpha+1} = \frac{m-r_s}{m}$. Also, $\delta_{\alpha} = \frac{z}{m} \frac{\partial}{\partial \tau} - \frac{\nu_0}{m} - s_{\alpha} k z$. By induction, (3.4) holds for $i = r_s$. We will prove (A.3) for $\alpha + 1$ and (3.5) for $j = m - r_s$.

Using (3.4) for $i = r_s$, a similar calculation gives (A.7) with i replaced by r_s . In the first sum, the term having the highest power of z is

$$z Q^{m+kms_{\alpha}} e^{m\tau+kms_{\alpha}\tau} \frac{\prod_{b < \{\frac{m-r_s}{k}\}} (\nu + bz)}{\prod_{b \leq \frac{m-r_s}{k}} (\nu + bz)} \mathbf{1}_{\frac{r_s-m}{k}} = O(1),$$

because $\frac{m-r_s}{k} = \{\frac{m-r_s}{k}\}$. In the second sum, the term having the highest power of z is

$$zQ^{r_s+kms_\alpha}e^{r_s\tau+kms_\alpha\tau}\frac{(\bar{\nu}+\frac{r_s}{m}z)}{\prod_{b=\{\frac{r_s}{m}\}}^{\frac{r_s}{m}}(\bar{\nu}+bz)}\mathbf{1}_{-r_s/m} = zQ^{r_s+kms_\alpha}e^{r_s\tau+kms_\alpha\tau}\mathbf{1}_{\frac{m-r_s}{m}},$$

because $0 < \frac{r_s}{m} < 1$. Note that

$$r_s + kms_\alpha = r_s + mq_s = ks = kms_{\alpha+1}.$$

We conclude that (A.3) holds for $\alpha + 1$. Further simplifying (A.7) for $i = r_s$ yields

$$\begin{aligned} & ze^{\tau\nu_0/z} \sum_{d \geq 0} \frac{Q^{dm+m+kms_\alpha}e^{dm\tau+(m+kms_\alpha)\tau}}{d!z^d} \frac{1}{\prod_{b=\{\frac{dm+m-r_s}{k}\}}^{\frac{dm+m-r_s}{k}}(\nu+bz)} \mathbf{1}_{-\frac{(dm+(m-r_s))}{k}} \\ & + ze^{\tau\nu_1/z} \sum_{d \geq 0} \frac{Q^{dk+r_s+kms_\alpha}e^{(dk+r_s+kms_\alpha)\tau}}{d!z^d} \frac{\prod_{b < \{\frac{dk-(m-r_s)}{m}\}}(\bar{\nu}+bz)}{\prod_{b \leq \frac{dk-(m-r_s)}{m}}(\bar{\nu}+bz)} \mathbf{1}_{-\frac{dk+(m-r_s)}{m}}, \end{aligned}$$

which in turn yields (3.5) for $j = m - r_s$ after removing the factor $Q^{sk}e^{sk\tau}$.

Case 3: $s_\alpha = \frac{s}{m}$, $s_{\alpha+1} = \frac{q_s+1}{k}$.

In this case $\tilde{s}_\alpha = \frac{m-r_s}{m}$, $\tilde{s}_{\alpha+1} = \frac{k-m+r_s}{k}$. Also, $\delta_\alpha = \frac{z}{k} \frac{\partial}{\partial \tau} - \frac{\nu_1}{k} - s_\alpha m z$. By induction, (3.5) for $j = m - r_s$ holds. We will prove (A.3) for $\alpha + 1$ and (3.4) for $i = k - m + r_s$.

Using (3.5) for $j = m - r_s$ we calculate

$$\begin{aligned} & (A.8) \\ & \delta_\alpha (Q^{kms_\alpha}e^{kms_\alpha\tau}z\partial_{\tilde{s}_\alpha}J_X) \\ & = ze^{\tau\nu_0/z} \sum_{d \geq 0} \frac{Q^{dm+m-r_s+kms_\alpha}e^{(dm+m-r_s+kms_\alpha)\tau}(\nu+\frac{dm+m-r_s}{k}z)}{d!z^d \prod_{b=\{\frac{dm+m-r_s}{k}\}}^{\frac{dm+m-r_s}{k}}(\nu+bz)} \mathbf{1}_{-\frac{(dm+m-r_s)}{k}} \\ & + ze^{\tau\nu_1/z} \sum_{d > 0} \frac{Q^{dk+kms_\alpha}e^{(dk+kms_\alpha)\tau}}{(d-1)!z^{d-1}} \frac{\prod_{b < \{\frac{dk-(m-r_s)}{m}\}}(\bar{\nu}+bz)}{\prod_{b \leq \frac{dk-(m-r_s)}{m}}(\bar{\nu}+bz)} \mathbf{1}_{-\frac{dk+m-r_s}{m}}. \end{aligned}$$

In the first sum, the term having the highest power in z is

$$z \frac{Q^{m-r_s+kms_\alpha}e^{(m-r_s+kms_\alpha)\tau}(\nu+\frac{m-r_s}{k}z)}{\prod_{b=\{\frac{m-r_s}{k}\}}^{\frac{m-r_s}{k}}(\nu+bz)} \mathbf{1}_{-\frac{m-r_s}{k}} = zQ^{m(q_s+1)}e^{m(q_s+1)\tau} \mathbf{1}_{\frac{k-m+r_s}{k}},$$

because $0 < \frac{m-r_s}{k} < 1$ and $m - r_s + kms_\alpha = m(q_s + 1) = kms_{\alpha+1}$. In the second sum, the term having the highest power in z is

$$zQ^{k+kms_\alpha}e^{(k+kms_\alpha)\tau} \frac{\prod_{b < \{\frac{k-m+r_s}{m}\}}(\bar{\nu}+bz)}{\prod_{b \leq \frac{k-m+r_s}{m}}(\bar{\nu}+bz)} \mathbf{1}_{-\frac{k+m-r_s}{m}} = O(1),$$

because $\frac{k-m+r_s}{m} > \frac{r_s}{m} > 0$. We conclude that (A.3) holds for $\alpha + 1$. Further simplifying (A.8) yields

$$\begin{aligned} & z e^{\tau \nu_0 / z} \sum_{d \geq 0} \frac{Q^{dm+kms_{\alpha+1}} e^{dm\tau+kms_{\alpha+1}\tau}}{d! z^d} \frac{\prod_{b < \frac{dm-(k-m+r_s)}{k}} (\nu + bz)}{\prod_{b \leq \frac{dm-(k-m+r_s)}{k}} (\nu + bz)} \mathbf{1}_{\frac{-dm+(k-m+r_s)}{k}} \\ & + z e^{\tau \nu_1 / z} \sum_{d \geq 0} \frac{Q^{dk+k+kms_{\alpha}} e^{(dk+k+kms_{\alpha})\tau}}{d! z^d} \frac{1}{\prod_{b=\frac{dk+(k-m+r_s)}{m}} (\bar{\nu} + bz)} \mathbf{1}_{\frac{-(dk+(k-m+r_s))}{m}}, \end{aligned}$$

which is easily seen to yield (3.4) for $i = k - m + r_s$, after using $k + kms_{\alpha} = kms_{\alpha+1} + (k - m + r_s)$ and removing the factor $Q^{kms_{\alpha+1}} e^{kms_{\alpha+1}\tau}$. This completes the induction, and the proof of the Corollary.

APPENDIX B. THE BI-GRADED EQUIVARIANT REDUCTION OF THE 2-TODA HIERARCHY

The 2-Toda lattice hierarchy was introduced by K. Ueno and K. Takasaki [23]. For the purpose of Gromov-Witten theory it is more convenient to introduce a hierarchy, which we also call 2-Toda, obtained from the 2-Toda lattice hierarchy by a certain infinitesimal lattice spacing limiting procedure (see [6]). From now on when we say 2-Toda we always mean the second one, not the original one.

B.1. Background on the 2-Toda hierarchy. The 2-Toda hierarchy consists of two sequences of flows on the manifold of pairs of Lax operators:

$$(B.1) \quad L = \Lambda + \sum_{i \geq 0} a_i \Lambda^{-i} \quad \text{and} \quad \bar{L} = Q e^v \Lambda^{-1} + \sum_{i \geq 0} \bar{a}_i \Lambda^i,$$

where Q is a fixed constant, a_i, \bar{a}_j, v are formal series in ϵ , whose coefficients are infinitely differentiable functions, v has no free term: $v = v^1(x)\epsilon + v^2(x)\epsilon^2 + \dots$, and Λ is a formal symbol which secretly should be thought as the shift operator $e^{\epsilon \partial_x}$, i.e., we demand that Λ and $u(x; \epsilon)$ satisfy the following commutation relation $\Lambda u(x; \epsilon) = u(x + \epsilon; \epsilon) \Lambda := \left(\sum_{k \geq 0} \frac{1}{k!} \epsilon^k \partial_x^k u(x; \epsilon) \right) \Lambda$.

The flows are defined by Lax type equations:

$$(B.2) \quad \epsilon \partial_{y_n} L = [(L^n)_+, L], \quad \epsilon \partial_{y_n} \bar{L} = [(\bar{L}^n)_+, \bar{L}], \quad n \geq 1,$$

$$(B.3) \quad \epsilon \partial_{\bar{y}_n} L = -[(\bar{L}^n)_-, L], \quad \epsilon \partial_{\bar{y}_n} \bar{L} = -[(\bar{L}^n)_-, \bar{L}], \quad n \geq 1,$$

where if M is a formal series in Λ and Λ^{-1} then we denote by M_+ (resp. M_-) the series obtained from M by truncating the terms with negative (resp. non-negative) powers of Λ .

Given a pair of Lax operators (B.1) we say that

$$\mathcal{P} = 1 + w_1(x; \epsilon) \Lambda^{-1} + w_2(x; \epsilon) \Lambda^{-2} + \dots$$

and

$$\mathcal{Q} = \bar{w}_0 + \bar{w}_1(\Lambda/Q) + \bar{w}_2(\Lambda/Q)^2 + \dots$$

form a pair of *dressing operators* if $L = \mathcal{P}\Lambda\mathcal{P}^{-1}$ and $\bar{L} = \mathcal{Q}Q\Lambda^{-1}\mathcal{Q}^{-1}$. According to [23], Proposition 1.4, the pair of Lax operators L and \bar{L} is a solution to the 2-Toda hierarchy if and only if there is a pair of dressing operators \mathcal{P} and \mathcal{Q} , called *wave operators*, such that

$$(B.4) \quad \epsilon \partial_{y_n} \mathcal{P} = -(L^n)_- \mathcal{P}, \quad \epsilon \partial_{y_n} \mathcal{Q} = (L^n)_+ \mathcal{Q},$$

$$(B.5) \quad \epsilon \partial_{\bar{y}_n} \mathcal{P} = -(\bar{L}^n)_- \mathcal{P}, \quad \epsilon \partial_{\bar{y}_n} \mathcal{Q} = (\bar{L}^n)_+ \mathcal{Q}, \quad n \geq 1.$$

Let us remark that the two sequences of time variables in [23], denoted there by x_n and y_n , correspond in our notations respectively to y_n/ϵ and $-\bar{y}_n/\epsilon$. The reason for the negative sign is that our definition of the flows (B.3) differs from the one in [23] by a negative sign.

Given a non-zero function $\tau(x, \mathbf{y}, \bar{\mathbf{y}}; \epsilon)$, where $\mathbf{y} = (y_1, y_2, \dots)$ and $\bar{\mathbf{y}} = (\bar{y}_1, \bar{y}_2, \dots)$, we define two operators $\mathcal{P} = 1 + w_1\Lambda^{-1} + w_2\Lambda^{-2} + \dots$ and $\mathcal{Q} = \bar{w}_0 + \bar{w}_1(\Lambda/Q) + \bar{w}_2(\Lambda/Q)^2 + \dots$, by

$$(B.6) \quad 1 + w_1\lambda^{-1} + w_2\lambda^{-2} + \dots = \frac{\exp\left(-\sum_{n=1}^{\infty} \frac{\lambda^{-n}}{n} \epsilon \partial_{y_n}\right) \tau(x, \mathbf{y}, \bar{\mathbf{y}}; \epsilon)}{\tau(x, \mathbf{y}, \bar{\mathbf{y}}; \epsilon)}$$

and

$$(B.7) \quad \bar{w}_0 + \bar{w}_1\lambda^{-1} + \bar{w}_2\lambda^{-2} + \dots = \frac{\exp\left(\sum_{n=1}^{\infty} \frac{\lambda^{-n}}{n} \epsilon \partial_{\bar{y}_n}\right) \tau(x + \epsilon, \mathbf{y}, \bar{\mathbf{y}}; \epsilon)}{\tau(x, \mathbf{y}, \bar{\mathbf{y}}; \epsilon)}.$$

The function $\tau(x, \mathbf{y}, \bar{\mathbf{y}}; \epsilon)$ is called τ -*function* of the 2-Toda hierarchy if the corresponding operators \mathcal{P} and \mathcal{Q} form a pair of wave operators, i.e., they satisfy equations (B.4)–(B.5).

Let us remark that our definitions of wave operators and τ -functions are slightly different from the ones in [23]. Namely, we define the wave operator \mathcal{Q} via the identity $\bar{L} = \mathcal{Q}(Q\Lambda^{-1})\mathcal{Q}^{-1}$, while in [23] the definition is $\bar{L} = \mathcal{Q}'\Lambda^{-1}(\mathcal{Q}')^{-1}$. On the other hand $Q\Lambda^{-1} = Q^{x/\epsilon}\Lambda^{-1}Q^{-x/\epsilon}$, therefore $\mathcal{Q}' = \mathcal{Q}Q^{x/\epsilon}$. Our excuse for departing from the standard definition is that we prefer to work with wave operators that admit a quasi-classical limit $\epsilon \rightarrow 0$. Note that if we put $\mathcal{Q}' = \bar{w}'_0 + \bar{w}'_1\Lambda + \bar{w}'_2\Lambda^2 + \dots$ and $\mathcal{Q} = \bar{w}_0 + \bar{w}_1(\Lambda/Q) + \bar{w}_2(\Lambda/Q)^2 + \dots$, then $\bar{w}'_i = \bar{w}_i Q^{x/\epsilon}$. This implies that if we define τ' the same way as τ except that in (B.7) we use \bar{w}'_i instead of \bar{w}_i then $\tau' = Q^{\frac{1}{2}((x/\epsilon)^2 - (x/\epsilon))} \tau$.

Let us introduce the following vertex operators:

$$\Gamma^{\pm} = \exp\left(\pm \sum_{n=1}^{\infty} (y_n/\epsilon) \lambda^n\right) \exp\left(\mp \sum_{n=1}^{\infty} \frac{\lambda^{-n}}{n} \epsilon \partial_{y_n}\right)$$

and $\bar{\Gamma}^\pm$ defined by the same formulas as Γ^\pm but with \bar{y}_n instead of y_n . Then according to [23], Theorem 1.7 and Proposition 1.6, the Lax operators $L = \mathcal{P}\Lambda\mathcal{P}^{-1}$ and $\bar{L} = \mathcal{Q}'\Lambda^{-1}(\mathcal{Q}')^{-1}$ form a solution of the 2-Toda hierarchy iff τ' satisfies the following HQEs:

$$\text{res}_{\lambda=\infty} \frac{d\lambda}{\lambda} \left(\lambda^{l-n} (\Gamma^+ \tau'_l) \otimes (\Gamma^- \tau'_{n+1}) - \lambda^{n-l} (\bar{\Gamma}^- \tau'_{l+1}) \otimes (\bar{\Gamma}^+ \tau'_n) \right) = 0,$$

where for every integer r we put $\tau'_r := \tau'(x + r\epsilon, \mathbf{y}, \bar{\mathbf{y}}; \epsilon)$. Substituting in the above HQEs the formula for τ' in terms of τ we get that $\tau(x, \mathbf{y}, \bar{\mathbf{y}}; \epsilon)$ is a τ -function iff the following HQEs hold:

(B.8)

$$\text{res}_{\lambda=\infty} \frac{d\lambda}{\lambda} \left(\lambda^{l-n} (\Gamma^+ \tau_l) \otimes (\Gamma^- \tau_{n+1}) - (Q\lambda^{-1})^{l-n} (\bar{\Gamma}^- \tau_{l+1}) \otimes (\bar{\Gamma}^+ \tau_n) \right) = 0.$$

B.2. The equivariant bi-graded reduction. According to the change of variables (1.4) and (1.5) we have $q_0^{0/k} = (\nu_0 - \nu_1)y_k$ and $q_0^{0/m} = (\nu_1 - \nu_0)\bar{y}_m$. Note that the shift of $q_0^{0/k}$ (resp. $q_0^{0/m}$) by $n\epsilon$ is equivalent to shifting y_k (resp. \bar{y}_m) by $\frac{n\epsilon}{\nu_0 - \nu_1}$ (resp. $\frac{n\epsilon}{\nu_1 - \nu_0}$). Motivated by Theorem 1.2 we ask the following

Question B.1. *What are the solutions L and \bar{L} of the 2-Toda hierarchy such that the corresponding τ -function has the form*

$$(B.9) \quad \tau(x, \mathbf{y}, \bar{\mathbf{y}}; \epsilon) = \mathcal{D}(y_1, \dots, y_k + \frac{x}{\nu_0 - \nu_1}, \dots, \bar{y}_1, \dots, \bar{y}_m + \frac{x}{\nu_1 - \nu_0}, \dots; \epsilon),$$

i.e., $(\nu_0 - \nu_1)\partial_x \tau = (\partial_{y_k} - \partial_{\bar{y}_m})\tau$?

This is equivalent to the following conditions on wave operators:

$$(B.10) \quad (\nu_0 - \nu_1)\partial_x \mathcal{P} = (\partial_{y_k} - \partial_{\bar{y}_m})\mathcal{P} \quad \text{and} \quad (\nu_0 - \nu_1)\partial_x \mathcal{Q} = (\partial_{y_k} - \partial_{\bar{y}_m})\mathcal{Q}.$$

We define the logarithms of the Lax operators L and \bar{L} by

$$\log L := \mathcal{P} \log \Lambda \mathcal{P}^{-1} := \epsilon \partial_x - (\epsilon \partial_x \mathcal{P}) \mathcal{P}^{-1}$$

and

$$\log \bar{L} := \mathcal{Q} \log (Q\Lambda^{-1}) \mathcal{Q}^{-1} := -\epsilon \partial_x + \log Q + (\epsilon \partial_x \mathcal{Q}) \mathcal{Q}^{-1}.$$

On the other hand from equations (B.10) we get

$$\epsilon \partial_x \mathcal{P} = \frac{1}{\nu_0 - \nu_1} (\epsilon \partial_{y_k} \mathcal{P} - \epsilon \partial_{\bar{y}_m} \mathcal{P}) = \frac{1}{\nu_0 - \nu_1} (-(L^k)_- \mathcal{P} + (\bar{L}^m)_- \mathcal{P})$$

and

$$\epsilon \partial_x \mathcal{Q} = \frac{1}{\nu_0 - \nu_1} (\epsilon \partial_{y_k} \mathcal{Q} - \epsilon \partial_{\bar{y}_m} \mathcal{Q}) = \frac{1}{\nu_0 - \nu_1} ((L^k)_+ \mathcal{Q} - (\bar{L}^m)_+ \mathcal{Q}).$$

Using our definition of logarithms of the Lax operators we write the above relations in the following form:

$$L^k + (\nu_1 - \nu_0) \log L = (L^k)_+ + (\bar{L}^m)_- + (\nu_1 - \nu_0) \epsilon \partial_x$$

and

$$\bar{L}^m + (\nu_0 - \nu_1) \log(Q^{-1}\bar{L}) = (L^k)_+ + (\bar{L}^m)_- + (\nu_1 - \nu_0)\epsilon\partial_x.$$

Now the description of the new hierarchy is the following. We define flows on the manifold of Lax operators

$$\mathcal{L} := \Lambda^k + \sum_{i=1}^k u_i \Lambda^{k-i} + \sum_{j=1}^{m-1} u_{k+j} \Lambda^{-j} + (Qe^v \Lambda^{-1})^m + (\nu_1 - \nu_0)\epsilon\partial_x.$$

Note that the equations $L^k + (\nu_1 - \nu_0) \log L = \mathcal{L}$ and $\bar{L}^m + (\nu_0 - \nu_1) \log(Q^{-1}\bar{L}) = \mathcal{L}$ have unique solutions of the types respectively $L = \Lambda + a_0 + a_1 \Lambda^{-1} + a_2 \Lambda^{-2} + \dots$ and $\bar{L} = Qe^v \Lambda^{-1} + \bar{a}_0 + \bar{a}_1 \Lambda + \bar{a}_2 \Lambda^2 + \dots$, where a_i and \bar{a}_j are formal series in ϵ whose coefficients are differential polynomials in $u_1, u_2, \dots, u_N := Qe^v$. The flows of the hierarchy are given by:

$$(B.11) \quad \epsilon\partial_{y_n} \mathcal{L} = [(L^n)_+, \mathcal{L}], \quad \epsilon\partial_{\bar{y}_n} \mathcal{L} = -[(\bar{L}^n)_-, \mathcal{L}], \quad n \geq 1.$$

One can check easily that this is a commuting set of flows. Also, by tracing back our argument, one can check that all solutions \mathcal{L} are given by

$$\mathcal{L} = (\mathcal{P}\Lambda^k\mathcal{P}^{-1})_+ + (\mathcal{Q}(Q\Lambda^{-1})^m\mathcal{Q}^{-1})_- + (\nu_1 - \nu_0)\epsilon\partial_x,$$

where \mathcal{P} and \mathcal{Q} are defined by formulas (B.6) and (B.7), for some function τ of the type (B.9) satisfying the bi-linear identities (B.8).

In order to check that we have an integrable hierarchy one needs to find a Hamiltonian formulation and prove the completeness of the flows. This could be done in the same way as in the article [13]. Another interesting problem is to prove that the Extended Bi-graded Toda Hierarchy (EBTH) defined in [5] is a non-equivariant limit of our hierarchy (B.11).

It is shown in [5] that EBTH is bi-hamiltonian, while the methods of E. Getzler [13] give only one Hamiltonian structure for (B.11). A natural question is whether the second Hamiltonian structure admits an equivariant deformation. A positive answer to the last question would be an indication that the big project of B. Dubrovin and Y. Zhang [11] admits a generalization in the context of equivariant quantum cohomology.

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